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A new finite-difference predictor-corrector method for fractional differential equations

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ABSTRACT

We present a new finite-difference predictor-corrector method (L1 - PCM) to solve nonlinear fractional differential equations (FDEs) along with its error and stability analysis. The method is further extended for systems of FDEs. The proposed method is applied to fractional version of chaotic system introduced by Bhalekar and Daftardar-Gejji to explore its rich dynamics. The proposed method is accurate, time-efficient and performs well even for very small values of the order of the derivatives.

1. Introduction

Fractional differential equations (FDEs) find applications to numerous phenomena in Science and Engineering, especially those which deal with memory effects. FDEs have become popular in modeling dynamical systems in several areas such as control theory, signal processing, electrical circuits, neuron modeling, viscoelasticity, and so on [1–4]. Transform methods such as Laplace transform, Fourier transform, Mellin transform etc. can be extended for linear FDEs, although solving nonlinear FDEs is a challenging task. Decomposition method introduced by Adomian [5] has been successfully employed to get analytical solution of non-linear FDEs [6–10]. One of the limitations of this method is tedious computations of Adomian polynomials. To circumvent this problem Daftardar-Gejji and Jafari [11] introduced a new decomposition method which will be referred as DGJ method in this article further. DGJ method is simple in its principles and does not involve discretization or rounding off errors. It can be easily implemented using computer algebra packages and does not involve calculations such as Adomian polynomials [5,12] or construction of homotopy as in case of homotopy perturbation method [13,14]. DGJ method is heavily used in the literature for solving a variety of problems [15–18]. For more details we refer the reader to a recent review of this method [19]. As decomposition methods yield local solutions around initial conditions, one has to resort to numerical methods for studying qualitative or long time behaviour of solutions of FDEs. Diethelm et al. [20] have extended Adams method to solve nonlinear FDEs numerically, known as Fractional Adams method (FAM) . This approach has been extensively used for simulations in the area of fractional ordered dynamical systems in the literature. Daftardar-Gejji et al. [21] have developed a new predictor-corrector method (NPCM) which is accurate and more time-efficient as compared to FAM. This method is also extended for FDEs with delay [22]. In the present paper, we introduce yet another accurate and time-efficient predictor-corrector method which involves discretization of the fractional derivatives.

The paper is organized as follows. In Section 2, preliminaries and notations are given. A new predictor-corrector method (L1-PCM) has been proposed in Section 3. Further error analysis and the stability analysis of L1-PCM is carried out in

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Section 4 and Section 5, respectively. In Section 6, the method is extended to solve system of FDEs. Numerous illustrative examples are presented in Section 7. In Section 8, L1-PCM is applied to fractional version of Bhalekar-Gejji system and relevant phase portraits are obtained.

2. Preliminaries

2.1. Definitions

Definition 1. Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f(t) \in C[a, b]$ is defined as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{1}$$

Definition 2. Caputo fractional derivative of order $\alpha > 0$ of a function $f \in C^m[a, b]$, $m-1 < \alpha \le m,\ m \in \mathbb{N}$ is defined as

$${}^cD_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left[\int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds \right] = I_a^{m-\alpha} D^m f(t), \tag{2}$$

where D^m is n-fold differential operator with $D^0 f(t) = f(t)$.

2.2. Properties of the fractional derivatives and integrals

1. Let $f \in C^m[a,b],\ m-1 < \beta \le m,\ m \in \mathbb{N}$ and $\alpha > 0$. Then
   (a) $I_a^\alpha({}^cD_a^\beta f(t)) = {}^cD_a^{\beta-\alpha} f(t)$, if $\alpha < \beta$.
   (b) $I_a^\beta({}^cD_a^\beta f(t)) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{\Gamma(k+1)} (t-a)^k$.
2. For $\alpha, \beta > 0$ and $f(t)$ sufficiently smooth,
   (a) if $\alpha \in \mathbb{N}$, then
   $${}^cD_a^\beta(I_a^\alpha f(t)) = I_a^{(\alpha-\beta)} f(t). \tag{3}$$
   (b) For $\alpha < \beta, m-1 \le \alpha < m, n-1 \le \beta < n$,
   $${}^cD_a^\beta(I_a^\alpha f(t)) = {}^cD_a^{\beta-\alpha} f(t) + \sum_{k=0}^{n-m} \frac{f^{(k)}(a)}{\Gamma(k+1+\alpha-\beta)} (t-a)^{k+\alpha-\beta}. \tag{4}$$
3. For $\alpha > 0, n \in \mathbb{N}$,
   $${}^cD_a^\alpha(D_a^n f(t)) = {}^cD_a^{n+\alpha} f(t). \tag{5}$$

2.3. DGJ method

Daftardar-Gejji and Jafari [11] introduced a new decomposition method (DGJ method) for solving functional equations of the form

$$y = g + N(y), \tag{6}$$

where g is a known function and $N(y)$ is non-linear operator from a Banach space $B \to B$.

Eq. (6) represents a variety of problems such as non-linear ordinary differential equations, integral equations, fractional differential equations, partial differential equations and systems of them.

In this method we assume that solution y of Eq. (6) is of the form:

$$y = \sum_{i=0}^{\infty} y_i. \tag{7}$$

The nonlinear operator is decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{k=0}^{i} y_k\right) - N\left(\sum_{k=0}^{i-1} y_k\right) \right\} \tag{8}$$

$$= \sum_{i=0}^{\infty} G_i, \tag{9}$$

where $G_0 = N(y_0)$ and $G_i = \left\{ N\left(\sum_{k=0}^{i} y_k\right) - N\left(\sum_{k=0}^{i-1} y_k\right) \right\},\ i \ge 1$.