Applied Mathematics and Computation 336 (2018) 433–453

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Fractional-order Legendre–Laguerre functions and their applications in fractional partial differential equations

H. Dehestani[a], Y. Ordokhani[a,*], M. Razzaghi[b]

[a] *Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran*
[b] *Department of Mathematics and Statistics, Mississippi State University, MS 39762, USA*

A R T I C L E I N F O

Keywords:
Fractional-order Legendre–Laguerre functions
Operational matrix of integration
Pseudo-operational matrix of integration
Fractional partial differential equation

A B S T R A C T

In this paper, we consider a new fractional function based on Legendre and Laguerre polynomials for solving a class of linear and nonlinear time-space fractional partial differential equations with variable coefficients. The concept of the fractional derivative is utilized in the Caputo sense. The idea of solving these problems is based on operational and pseudo-operational matrices of integer and fractional order integration with collocation method. We convert the problem to a system of algebraic equations by applying the operational matrices, pseudo-operational matrices and collocation method. Also, we calculate the upper bound for the error of integral operational matrix of the fractional order. We illustrated the efficiency and the applicability of the approach by considering several numerical examples in the format of table and graph. We also describe the physical application of some examples.

1. Introduction

Fractional calculus has a long history and it has been used in various fields of science, engineering, applied mathematics and economics [1–5]. Nowadays, mathematicians observe fractional differential equations in different real world problems such as physical, chemical, fluid mechanics, mathematical biology, electro chemistry and other sciences [6–11]. Many researchers have utilized various numerical methods to solve these equations. For instance, Ertrk and Momani [12], discussed the systems of fractional differential equations by using differential transform method, Jafari et al. [13], applied Legendre wavelets for solving fractional differential equations, Rahimkhani et al. [14], introduced fractional-order Bernoulli wavelets for solving fractional differential equations. In [15], the authors solved a multi-order fractional differential equation by using Adomian decomposition, Babolian and Shamloo [16], have solved Volterra integral and integro-differential equations of convolution type by using operational matrices of piecewise constant orthogonal functions, Keshavarz et al. [17], introduced Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. For more information about this topic, refer to [18–22].

In this paper, we introduce new fractional-order Legendre–Laguerre functions for solving time-space fractional partial differential equations. There exist several analytical and numerical methods for solving various classes of fractional partial differential equations, which express some of them: Yildirim and Koak [23], discussed the space-time fractional advection dispersion equation by using of Homotopy perturbation method. Uddin and Haq [24], introduced RBFs approximation

* Corresponding author.
E-mail address: ordokhani@alzahra.ac.ir (Y. Ordokhani).

https://doi.org/10.1016/j.amc.2018.05.017
0096-3003/

method for time fractional partial differential equations. Jin et al. [25], used the Galerkin finite element method for a multi-term time-fractional diffusion equation. Chen et al. [26], applied wavelet method for a class of fractional convection-diffusion equations with variable coefficients. Saadatmandi and Dehghan [27], solved the space fractional diffusion equation by tau approach. Bhrawy et al. [28], applied spectral tau algorithm based on jacobi operational matrix for solving time fractional diffusion-wave equations. In addition, Dehghan et al. [29], utilized the variational iteration method for solving the telegraph and fractional telegraph equations. Zhou and Xu [30] introduced Chebyshev wavelets collocation method for solving the time-fractional convection diffusion equations with variable coefficients. Also, readers who are interested in learning more about this subject can refer to [31–40].

The structure of this article is as follows. The following section contains concepts of the fractional calculus theory. In Section 3, we introduce fractional-order Legendre–Laguerre functions, their features and function approximations. Section 4 is devoted to the integral operational and pseudo-operational matrices with integer and fractional order for fractional-order Legendre and Laguerre functions. In Section 5, we construct an algorithm for solving linear and nonlinear time-space fractional partial differential equations. Error analysis is performed in Section 6. In Section 7, we present some numerical results, which demonstrate the accuracy of the proposed numerical scheme. Also, a conclusion is given in Section 8.

2. Preliminaries

We give some basic concepts of the fractional calculus theory, which are used further in this paper [1–5].

Definition 1. The Riemann–Liouville fractional integral operator of order $\nu \geq 0$ is defined as

$$I^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \qquad x \geq 0, \qquad \nu \geq 0,$$
$$I^0 f(x) = f(x).$$

The properties of the operator I^{ν} can be found in [5]. Below, we considered a number of them, for $\gamma > -1$:

- $I^{\nu} I^{\gamma} f(x) = I^{\nu+\gamma} f(x)$,
- $I^{\nu} I^{\gamma} f(x) = I^{\gamma} I^{\nu} f(x)$,
- $I^{\nu} x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\nu+\gamma+1)} x^{\nu+\gamma}$.

Definition 2. The fractional derivative of order ν in the Caputo sense is defined as

$$D^{\nu} f(x) = I^{m-\nu}(D^m f(x)) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} f^{(m)}(t)dt,$$
$$m-1 < \nu \leq m, \qquad m \in \mathbb{N}, \qquad x > 0,$$

which has the following properties:

- $D^{\nu} C = 0, \qquad (C \textit{ is a constant})$
- $D^{\nu} I^{\nu} f(x) = f(x)$,
- $I^{\nu} D^{\nu} f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0) \frac{x^i}{i!}$,
- $D^{\nu} x^{\gamma} = \begin{cases} 0, & \gamma \in N_0, \quad \gamma < \nu, \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\nu)} x^{\gamma-\nu}, & otherwise. \end{cases}$

3. Fractional-order functions

To solve fractional partial differential equations, we introduce fractional-order Legendre–Laguerre functions.

3.1. Fractional-order Legendre functions

Consider the fractional-order shifted Legendre functions $P_m^{\alpha}(x)$, on the interval [0,1] as [41]

$$P_0^{\alpha}(x) = 1, \qquad P_1^{\alpha}(x) = 2x^{\alpha} - 1,$$
$$P_{m+1}^{\alpha}(x) = \frac{(2m+1)(2x^{\alpha}-1)}{m+1} P_m^{\alpha}(x) - \frac{m}{m+1} P_{m-1}^{\alpha}(x), \qquad m = 1, 2, 3, \ldots. \tag{1}$$

The fractional-order Legendre functions are a particular solution of the normalized eigenfunctions of the singular Sturm–Liouville problem [41]

$$((x - x^{1+\alpha}) P_m'^{\alpha}(x))' + \alpha^2 m(m+1) x^{\alpha-1} P_m^{\alpha}(x) = 0, \qquad x \in (0, 1).$$