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Approximation of the modified error function

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ABSTRACT

In this article, we obtain explicit approximations of the modified error function introduced in *Cho and Sunderland* (1974), as part of a Stefan problem with a temperature-dependent thermal conductivity. This function depends on a parameter δ, which is related to the thermal conductivity in the original phase-change process. We propose a method to obtain approximations, which is based on the assumption that the modified error function admits a power series representation in δ. Accurate approximations are obtained through functions involving error and exponential functions only. For the special case in which δ assumes small positive values, we show that the modified error function presents some characteristic features of the classical error function, such as monotony, concavity, and boundedness. Moreover, we prove that the modified error function converges to the classical one when δ goes to zero.

1. Introduction

Phase-change processes are present in a broad variety of natural, technological and industrial situations [1,5,9,17,24,25,28]. Modeling them properly is then crucial for understanding or predicting the evolution of many physical processes. One common assumption when modeling phase-change processes is to consider constant thermophysical properties. Nevertheless, it is known that certain materials present properties which seem to obey other laws. Recently, some models including variable latent heat, density, melting temperature or thermal conductivity have been proposed in [2,4,21,22,34,36,44].

In this sense, in 1974, Cho and Sunderland presented a similarity solution for a Stefan problem in which the thermal conductivity is a linear function of the temperature distribution [13]. It is well known that similarity solutions to Stefan problems with constant coefficients can be expressed in terms of the error function erf,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\xi^2) d\xi \quad x > 0. \tag{1}$$

In contrast to this, the solution obtained by Cho and Sunderland involves another function, which they have called *modified error function*. It was defined as the solution to a nonlinear boundary value problem, and its existence and uniqueness

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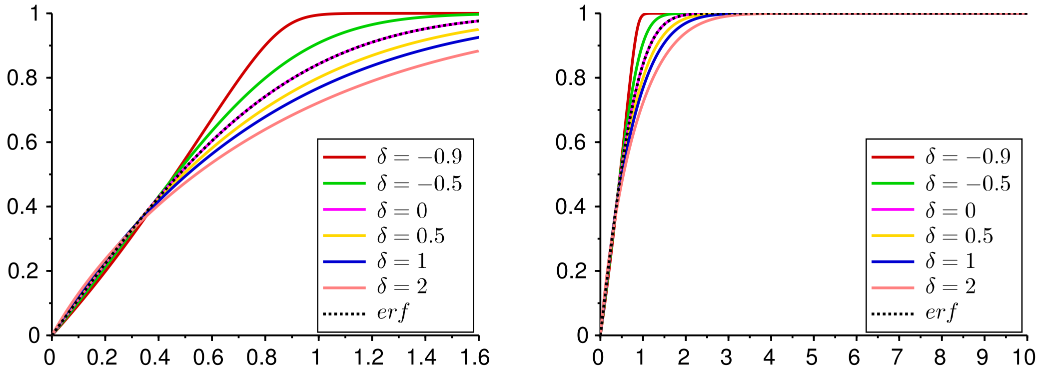

(a) $x \in [0, 1.6]$ (b) $x \in [0, 10]$

Fig. 1. Modified error function Φ_δ for *delta* $= -0.9, -0.5, 0, 0.5, 1, 2$ over different domains.

was recently proved in [10] for thermal conductivities with moderate variations. In spite of the latter, the modified error function was widely used for solving diffusion problems [8,14,23,29,32,37,40], even before it was formally introduced by Cho and Sunderland in 1974 [15,45].

When phase-change processes come from technological or industrial problems, not only appropriate models are required but also their solutions (or, at least, some properties of them). Sometimes, when explicit solutions are not known, models are solved through numerical methods which are tested with experimental data. When the latter are not available, one common practice is to test numerical methods by applying them to another problem whose explicit solution is known. Thus, having explicit solutions to models for phase-change processes is sometimes quite useful. Many works have been done in this direction, see for example [3,6,7,11,12,16,18–20,26,27,30,31,35,38,39,41–44,46]. Regarding the model in [13], explicit solutions are not known yet. Aiming to make a contribution in this sense, the main goal of this article is to propose some approximations of the modified error function.

In order to present our ideas clearly, we briefly recall how the modified error function arises from the original phase-change process. For simplicity, we consider the case of a one-phase melting problem for a semi-infinite slab with phase-change temperature T_m, whose boundary $x = 0$ is maintained at a constant temperature $T_\infty > T_m$. For this case, the thermal conductivity from Cho and Sunderland is

$$k(T) = k_0 \left\{ 1 + \delta \left(\frac{T - T_\infty}{T_m - T_\infty} \right) \right\}, \tag{2}$$

where $k_0 > 0$ is the thermal conductivity at $x = 0$, and δ is some dimensionless parameter. Since $T = T_m$ at the free boundary, $\delta > -1$ becomes a necessary condition to assure the thermal conductivity is positive when $x = s(t)$. When the temperature distribution is assumed to be in the form $T(x,t) = A + B\Phi_\delta\left(\frac{x}{2\sqrt{\alpha_0 t}}\right)$,[1] for A and B constant, one obtains that Φ_δ can be found by solving the following nonlinear boundary value problem (see details in [13]):

$$[(1 + \delta y(x))y'(x)]' + 2xy'(x) = 0 \quad 0 < x < +\infty \tag{3a}$$

$$y(0) = 0 \tag{3b}$$

$$y(+\infty) = 1. \tag{3c}$$

The solution Φ_δ to this problem is the already mentioned *modified error function*. Some plots for Φ_δ are shown in Fig. 1. They were obtained by numerically solving problem (3) for $\delta = -0.9, -0.5, 0, 0.5, 1, 2$. For the special case in which $\delta = 0$, which corresponds to a constant thermal conductivity, one finds that the modified error function coincides with the classical one. The coincidence is stronger than that shown from the numerical computations, since it can be easily proved that the error function is the only solution to problem (3) when $\delta = 0$. As δ moves away from zero, the modified error function differs more and more from the classical one. Nevertheless, both functions seem to share some properties (such as non-negativity, boundedness and rapid convergence to 1 when $x \to +\infty$). Moreover, when $\delta > 0$, the modified error function seems to be increasing and concave, as the error function is. Observe that $-1 < \delta < 0$ is related to thermal conductivities

[1] α_0 is the coefficient of diffusion at $t = 0$.