Applied Mathematics and Computation 336 (2018) 481–489

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical analysis for Navier–Stokes equations with time fractional derivatives[☆]

Jun Zhang[a], JinRong Wang[b,c,*]

[a] Computational mathematics research center, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, PR China
[b] Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, PR China
[c] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, PR China

ARTICLE INFO

Keywords:
Navier–Stokes equations
Caputo fractional derivative
Finite difference
Legendre-spectral method
Error estimate

ABSTRACT

In this article, we study numerical approximation for a class of Navier–Stokes equations with time fractional derivatives. We propose a scheme using finite difference approach in fractional derivative and Legendre-spectral method approximations in space and prove that the scheme is unconditionally stable. In addition, the error estimate shows that the numerical solutions converge with the order $\mathcal{O}(\Delta t^{2-\alpha} + \Delta t^{-\alpha} N^{1-s})$, $0<\alpha<1$ being the order of the fractional derivative in time. Numerical examples are illustrated to verify the theoretical results.

1. Introduction

This paper is devoted to investigate a numerical scheme for the approximation of following time-fractional Navier–Stokes equations (TFNSE):

$$D_t^{\alpha}\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in} \quad \Omega\times[0,T], \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \quad \text{in} \quad \Omega\times[0,T], \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \tag{1.2}$$

$$u = 0, \quad \text{on} \quad \partial\Omega\times[0,T], \tag{1.3}$$

where $\alpha\in(0,1)$ and D_t^{α} denotes the Caputo fractional derivative, $\Omega\subset\mathbf{R}^2$ is a bound domain, $\mathbf{u}=(u_1,u_2)$ represents the velocity, and p which represents the pressure. $\nu>0$ is the viscosity coefficient and $\mathbf{f}=(f_1,f_2)$ is an external force.

TFNSE have been studied extensively due to their applications in polymer physics, electrochemistry of corrosion and many other physical processes. Some recent theoretical and numerical results show that the classical diffusion equation can not describe the phenomena in heterogeneous porous media with fractal characteristics, but fractional differential equations are effective in simulating anomalous diffusion process. Fractional derivatives are found to be quite flexible in describing

☆ This work is partially supported by National Natural Science Foundation of China (11661016) and Guizhou University of Finance and Economics (No.2017XZD01).

* Corresponding author at: Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, PR China.
E-mail addresses: zj654440@163.com (J. Zhang), wangjinrong@gznc.edu.cn, sci.jrwang@gzu.edu.cn (J. Wang).

https://doi.org/10.1016/j.amc.2018.04.036
0096-3003/

complex dynamics in viscoelastic fluids, and the hydrodynamic model with fractional derivative can eliminate the shortage of continuous flow hypothesis [1–5]. Based on the experimental data, He [6] proposed a fractional partial differential equation for porous media percolation. Liu et al. [7–11] presented a series of numerical methods to solve the viscoelastic fluid problem with fractional derivative.

The problem (1.1)–(1.3) turns into classical Navier–Stokes equations when $\alpha = 1$, the related numerical methods of which have been found in [12–18] and the references therein. Recently, El-Shahed and Salem [19] obtained the analytical solution for TFNSE by using Laplace, Fourier Sine transforms, and finite Hankel transforms. Carvalhoneto and Planas [20] studied existence and uniqueness properties of local mild solutions to the TFNSE. It is remarkable that Zhou and Peng [21,22] investigated the existence and uniqueness of global and local mild solutions of TFNSE.

In spite of basic theory analysis for TFNSE, some recent contributions focus on using numerical methods to approximate the solutions of TFNSE. For example, Ganji et al. [23], Kumar et al. [24] have introduced a homotopy perturbation method to simulate differential equations of Caputo fractional order arising in fluid mechanics. Kumar et al. [25] have discussed the approximated analytical solutions of TFNSE by using modified fractional Laplace decomposition method. In [26], a mixed finite element method was proposed and analyzed, under some certain assumptions on the $\mathbf{u}$ and $\mathbf{f}$, they derived some error estimates for numerical schemes. Momani and Obibat [27] proposed a Adomian decomposition method for TFNSE in a tube.

On the other hand, many numerical techniques are used to investigate fractional differential equations. Lin and Xu [28] proposed a finite difference/Legendre collocation spectral method to approximate Caputo fractional equation, and their numerical method leads to 2-α order accuracy in time and spectral accuracy in space. Li and Xu [29] constructed a time-space spectral method for fractional partial differential equations. Bhrawy et al. [30] presented a shifted Legendre collocation method to solve a time-space fractional Burgers' equation in Caputo sense. Bhrawy et al. [31,32] introduced a Legendre spectral tau method to investigate the time-space fractional diffusion equation.

At present, there is still a lack of effective numerical method to investigate TFNSE. In this article, we introduce and discuss a efficient numerical scheme to solve TFNSE. The proposed scheme is performed combining finite difference scheme for fractional derivative and spectral discretization for spatial variable. A detailed analysis of the numerical scheme is provided for both stability and error estimate. Under the appropriate assumptions, our rigorous analysis result show that the scheme is unconditionally stable, and that the convergent order is $\mathcal{O}(\Delta t^{2-\alpha} + \Delta t^{-\alpha} N^{1-s})$, where Δt, N and s are, respectively step of time, polynomial degree, and regularity of $\mathbf{u}$. At last, some numerical experiments are conducted to confirm the theoretical claims.

The rest of the article is organized in following way. Section 2 introduces a scheme for fractional derivative. In Section 3, we discuss a error estimate of the full discrete scheme. In Section 4, we present some numerical experiments to illustrate the validity of the numerical method. The conclusions of this paper is given in Section 5.

2. Stability for semi-discretization fractional Navier–Stokes equations

We now introduce some of the notations in this article. We use the standard notation $L^2(\Omega)$, $H^s(\Omega)$, and $H_0^s(\Omega)$ to denote the usual Sobolev spaces. We use $\|\cdot\|$ to denote the norm in $L^2(\Omega)$, and $(\cdot, \cdot)$ to denote the scalar product in $L^2(\Omega)$. The dual space of $H_0^1(\Omega)$ will be denote by $H^{-1}(\Omega)$, and the duality between them will be denoted by $\langle ., . \rangle$. We also denote the following spaces

$$X = \{q \in L^2(\Omega) : \int_\Omega q d\Omega = 0\}. \tag{2.1}$$

And a trilinear operator $B(\cdot, \cdot)$ by

$$B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}), \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in H_0^1(\Omega). \tag{2.2}$$

Therefore, we have

$$B(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \ \mathbf{u}, \mathbf{v}, \in H_0^1(\Omega). \tag{2.3}$$

From [33] we have the following inequality:

$$\|B(\mathbf{u}, \mathbf{v}, \mathbf{w})\| \le C_0 \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\|, \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in H_0^1(\Omega). \tag{2.4}$$

In this context, we consider the discrete Eq. (1.1) and (1.2) in time. First, we introduce a scheme to discrete the Caputo derivative. For a given number $T > 0$ and integer M, let $t_n = n\Delta t, n = 0, 1, \ldots, M-1$, where $\Delta t = T/M$. From [28,34], we know that the Caputo derivative $D_t^\alpha \mathbf{u}$ can be approximated by

$$\frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j \frac{\mathbf{u}(t_{n+1-j}) - \mathbf{u}(t_{n-j})}{\Delta t^\alpha}, \tag{2.5}$$

where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$.