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Multiphase segmentation for simultaneously homogeneous and textural images

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ABSTRACT

Segmentation remains an important problem in image processing. For homogeneous images containing only piecewise smooth information, a number of important models have been developed and refined over the past several decades. However, these models often fail when applied to the substantially larger class of natural images that simultaneously contain regions of homogeneity and non-homogeneity such as texture. This work introduces a bi-level constrained minimization model for simultaneous multiphase segmentation of images containing both homogeneous and textural regions. We develop novel norms defined in different functional Banach spaces for the segmentation which results in a non-convex minimization. Finally, we develop a generalized notion of segmentation delving into approximation theory and demonstrating that a more refined decomposition of these images results in multiple meaningful components. Both theoretical results and demonstrations on natural images are provided.

1. Introduction

Image segmentation remains at the forefront of issues in computer vision and image processing and an abundance of approaches have been developed to solve a wide range of problems; see, for example, [1,2]. The goal of image segmentation is to decompose the image domain into a montage of meaningful components. This has lead to breakthroughs in a number of research areas such as medical imaging [3], astronomical imaging [4], and biometric recognition [5–7].

Segmentation methods can be broadly characterized by the class of target images for which they are intended. Often, the two primary image classes considered are (1) homogeneous images that are assumed to be piecewise smooth (e.g. a sunset) and (2) textural images that contain repeated patterns of similar intensity (e.g. zebra stripes). Many such methods have been suggested, including approaches based on the intensity of pixels [8–10] and others based on curve evolution [11–14]. For homogeneous images in particular, the classical approach to segmentation is based on active contours [15]. Given an image f on a bounded domain $\Omega \subset \mathbb{R}^2$, contours are driven to object boundaries by internal and external forces in the functional

$$\inf_{C(s)} \left\{ \alpha \int_0^1 \left| C'(s) \right|^2 ds + \beta \int_0^1 \left| C''(s) \right| ds - \lambda \int_0^1 |\nabla f(C(s))|^2 ds \right\} \tag{1}$$

with a curve $C(s) : [0, 1] \to \mathbb{R}^2$ and positive parameters α, β and λ.

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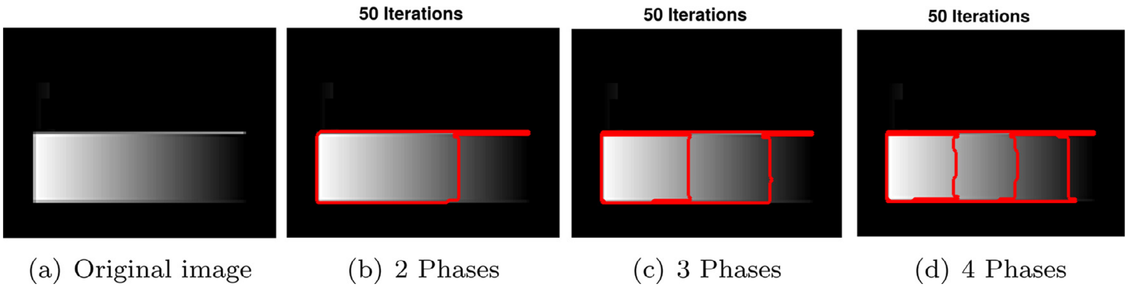

Fig. 1. The original image (a) is segmented by Bae et al. [42] with two phase (b), three phase (c) and four phase segmentation (with the same notation for parameters $n = 4, s = 0.001, \delta = 0.1$). Using multiple phases allows us to recover portions of the image with different illumination (gradient change).

Under the classical model $f(\boldsymbol{x}) = u(\boldsymbol{x}) + \epsilon(\boldsymbol{x})$ with $\boldsymbol{x} \in \Omega$, Mumford and Shah [16] proposed a solution by minimizing the energy functional

$$\inf_{u,C} \left\{ \int_\Omega \big(f(\boldsymbol{x}) - u(\boldsymbol{x}) \big)^2 d\boldsymbol{x} \; + \; \nu \int_{\Omega \backslash C} |\nabla u(\boldsymbol{x})|^2 d\boldsymbol{x} + \mu |C| \right\}. \tag{2}$$

However, this piecewise smooth Mumford–Shah model is NP-hard due to the Hausdorff 1-dimensional measure $\mathcal{H}^1(\mathbb{R}^2)$. A simplified version for image segmentation can be written as

$$\inf_{[c_n]_{n=1}^N, [\Omega_n]_{n=1}^N} \left\{ \sum_{n=1}^N \int_\Omega \big(f(\boldsymbol{x}) - c_n \big)^2 \mathbf{1}_{\Omega_n}(\boldsymbol{x}) d\boldsymbol{x} \; + \; \frac{\mu}{2} \sum_{n=1}^N \int_\Omega |\nabla \mathbf{1}_{\Omega_n}(\boldsymbol{x})| d\boldsymbol{x} \right\}. \tag{3}$$

where f is assumed to be piecewise constant. This simplified model is similar in form to the Potts model [17] developed decades earlier. Rudin et al. [18] proposed an alternative, more computationally efficient version of the model in (2) that preserves sharp edges in the restored image. These advantages led to numerous extensions including examination in different functional spaces, [19–24], versions involving higher-order derivatives [25–28], mean curvature [29], Euler's elastica [30,31], total variation of the first and second order derivatives [32], and higher-order PDEs for diffusion solved by directional operator splitting schemes [33]. Various techniques have been proposed for solving the convex optimization including Chambolle's projection [34], the splitting Bregman method [35], and iterative shrinkage/thresholding (IST) algorithms [36–38]. In 2010, Wu et al. [39] proved the equivalence between the augmented Lagrangian method (ALM), dual methods, and the splitting Bregman method.

Letting $p_n(\boldsymbol{x})$ denote the indicator function $\mathbf{1}_{\Omega_n}(\boldsymbol{x})$ with $\boldsymbol{x} \in \Omega$, the equation in (3) can be rewritten as the non-convex constrained minimization

$$\min_{\vec{c}, \vec{p}} \left\{ \frac{\mu}{2} \sum_{n=1}^N \| \nabla p_n \|_{L_1} + \sum_{n=1}^N \langle (f - c_n)^2, p_n \rangle_{L_2} \text{s.t.} \sum_{n=1}^N p_n(\boldsymbol{x}) = 1, \, p_n(\boldsymbol{x}) \in \{0, 1\} \right\} \tag{4}$$

where $\vec{c} = [c_n]_{n=1}^N$, $\vec{p} = [p_n]_{n=1}^N$. When $N = 2$, this becomes the celebrated Chan and Vese model [11]. Brown et al. [40,41] provide a convex relaxation of (4) by relaxing the binary set to $p_n(\boldsymbol{x}) \in [0, 1]$ and Bae et al. [42] solve this relaxed version via a smoothed primal-dual method; see [43–46] for details. The advantage of the multiphase segmentation is illustrated in Fig. 1 using the smoothed primal-dual method in [42] to recover an object under a spectrum of illumination. Outside the class of homogeneous images, the segmentation and analysis of images consisting of texture remains a challenging problem due to the inherent oscillatory nature of such information. Proposed methods include those based on texture descriptors [47,48], histogram metrics [49], and finding other meaningful features in an observed image for classification [50]. Among the most popular approaches is the vector-valued Chan–Vese model for texture segmentation with a Gabor filter [51] whose convex relaxed version is defined in [40,52]. This can be seen as a generalized version of the two-phase piecewise constant Mumford and Shah model for a vector valued image $\vec{f} = [f_m]_{m=1}^M$ and constant vector $\vec{c}_1 = [c_{1m}]_{m=1}^M$, $\vec{c}_2 = [c_{2m}]_{m=1}^M$. The resulting minimization becomes convex by relaxing the binary constraint to $p(\boldsymbol{x}) \in [0, 1]$ [40].

Though these techniques have seen much success in their respective domains, they fall short on many natural images like fingerprints and stem cell imaging that **s**imultaneously contain both **h**omogeneous and **t**extural (SHT) regions. See Fig. 2 for one such example where textural regions appear in each phase due to the inability of the Mumford–Shah model to measure such aspects. The primary purpose of our work here is to develop a technique capable of capturing the entirety of this homogeneous and textural information. An attempt at this kind of segmentation was provided by Liu et al. [53] but relied on harmonic analysis. Importantly, our work here can be viewed instead as a decomposition of the original image into well-separated, meaningful components which approximates the image in functional space. This approach to the inverse problem allows us to obtain a piecewise constant component as well as sparse directional information. Our SHT models accurately capture the textural boundaries within an image and also allows for the separation of this information from the remainder of the image. Furthermore, on homogenous-only images, our SHT models compare favorably with