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Cramer's rule for a system of quaternion matrix equations with applications[☆]

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ABSTRACT

In this paper, we investigate Cramer's rule for the general solution to the system of quaternion matrix equations

$$A_1XB_1 = C_1, \ A_2XB_2 = C_2,$$

and Cramer's rule for the general solution to the generalized Sylvester quaternion matrix equation

$$AXB + CYD = E,$$

respectively. As applications, we derive the determinantal expressions for the Hermitian solutions to some quaternion matrix equations. The findings of this paper extend some known results in the literature.

1. Introduction

Throughout, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1, \ a_0, a_1, a_2, a_3 \ \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, the identity matrix with the appropriate size by I. The symbols $\mathcal{R}_r(A)$, $\mathcal{N}_r(A)$, $\mathcal{R}_l(A)$ and $\mathcal{N}_l(A)$ stand for the right column space, the right null space, the left row space and the left null space of a matrix $A \in \mathbb{H}^{m\times n}$, respectively. The Moore–Penrose inverse of $A \in \mathbb{H}^{m\times n}$, denoted by $A\dagger$, is the unique matrix X satisfying the Penrose equations

$$AXA = A, \ XAX = X, \ (AX)^* = AX, \ (XA)^* = XA.$$

L_A and R_A stand for the two projectors $L_A = I - A^\dagger A, R_A = I - AA^\dagger$ induced by A.

Research on the system of matrix equations

$$\begin{cases} A_1XB_1 = C_1 \\ A_2XB_2 = C_2 \end{cases} \tag{1.1}$$

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has been actively ongoing for many years. For instance, Navarra [1] derived a necessary and sufficient condition for the existence of a common solution to the pair of linear matrix equations. Mitra [2] and [3] provided the necessary and sufficient conditions for the existence and a representation of the general solution of (1.1). Peng et al. [4] considered an efficient algorithm for the least-squares reflexive solution of (1.1). Some other results relate to the consistent conditions as well as the expressions of the general solution of (1.1) can be founded in [5] and [6]. Meanwhile, many studies have also been done on the solution to the generalized Sylvester equation

$$AXB+CYD=E. \tag{1.2}$$

In 1987, Chu [7] studied the compatibility and the least norm solution of (1.2) by using the general singular value decomposition (GSVD). In 1998, Xu and Zheng [8] derived the least square solution and the symmetric (anti-symmetric) solution of $AXA^*+CYC^*=F$ by using the canonical correlation decomposition (CCD). In 2006, Liao et al. [9] studied the least square solution of (1.2) with the least norm by combining CCD and GSVD. Some other results can be found in [10–19]. It follows from the above results that there are some special relationships between the consistence of (1.1) and (1.2): the generalized Sylvester equation (1.2) is consistent if and only if

$$\begin{cases} R_CAXB=R_CE \\ AXBL_D=EL_D \end{cases}$$

is consistent with respect to X. By this special relationship, Wang [20] derived the necessary and sufficient conditions for the existence and expressions of the general solution to (1.1) and (1.2) over arbitrary regular rings with identity, respectively.

Cramer's rule is often used as a basic method to express the unique solution to some consistent matrix equations or the best approximate solution to some inconsistent matrix equations. In 1970, Steve Robinson [21] gave an elegant proof of Cramer's rule over the complex number field. After that, many authors [22–29] studied the generalized inverses and solutions of some restricted equations by Cramer's rules. However, we can not generalize the existing Cramer's rules to the quaternion skew field directly, since the multiplication of quaternions is not commutative and there are some differences between the determinants of quaternion matrix and complex matrix. In 2008, Kyrchei [30] defined the row and column determinants of a square matrix over the quaternion skew field. Moreover, he proved Cramer's rules for the unique solution, the minimum norm least square solution of some quaternion matrix within the framework of the theory of the row and column determinants in [31–37], respectively. In addition, we [38–41] derived Cramer's rule for the unique solution of some restricted quaternion matrix equations. To our best knowledge, there has been little research on expressing the general solution of (1.1) and (1.2) by Cramer's rules.

Motivated by the work mentioned above, and keep the interesting of Cramer's rules theory, we in this paper aim to consider Cramer's rules for the general solution to the system (1.1) and the generalized Sylvester Eq. (1.2) over the quaternion skew field, respectively. The paper is organized as follows. In Section 2, we derive some condensed Cramer's rules for the general solution of (1.1). As applications, we derive Cramer's rules for the Hermitian solutions to some quaternion matrix equations. In Section 3, we derive some Cramer's rules for the general solution of (1.2). In Section 4, we show a numerical example to illustrate the main results. To conclude this paper, we propose some further research topics in Section 5.

2. Cramer's rule for the general solution of (1.1)

Unlike multiplication of real or complex numbers, multiplication of quaternions is not commutative. Many authors [42–46] had tried to give the definitions of the determinants of a quaternion matrix. Unfortunately, by their definitions it is impossible for us to give a determinantal representation of an inverse of matrix. In 2008, Kyrchei [30] defined the row and column determinants of a square matrix over the quaternion skew field as follows. Suppose S_n is the symmetric group on the set $I_n=\{1,\ldots,n\}$.

Definition 2.1 (Definition 2.4–2.5 [30]). (1) The ith row determinant of $A=\left(a_{ij}\right)\in\mathbb{H}^{n\times n}$ is defined by

$$\mathrm{rdet}_i A=\sum_{\sigma\in S_n}(-1)^{n-r}a_{ii_{k_1}}a_{i_{k_1}i_{k_{1+1}}}\ldots a_{i_{k_1+l_1}i}\ldots a_{i_{k_r}i_{k_r+1}}\ldots a_{i_{k_r+l_r}i_{k_r}}$$

for all $i=1,\ldots,n$. The elements of the permutation σ are indices of each monomial. The left-ordered cycle notation of the permutation σ is written as follows:

$$\sigma=\left(ii_{k_1}i_{k_1+1}\ldots i_{k_1+l_1}\right)\left(i_{k_2}i_{k_2+1}\ldots i_{k_2+l_2}\right)\ldots\left(i_{k_r}i_{k_r+1}\ldots i_{k_r+l_r}\right).$$

The index i opens the first cycle from the left and other cycles satisfy the following conditions, $i_{k_2}<i_{k_3}<\ldots<i_{k_r}$ and $i_{k_t}<i_{k_t+s}$ for all $t=2,\ldots,r$ and $s=1,\ldots,l_t$. (2) The jth column determinant of $A=\left(a_{ij}\right)\in\mathbb{H}^{n\times n}$ is defined by

$$\mathrm{cdet}_j A=\sum_{\tau\in S_n}(-1)^{n-r}a_{j_{k_r}j_{k_r+l_r}}\ldots a_{j_{k_r+1}j_{k_r}}\ldots a_{jj_{k_1+l_1}}\ldots a_{j_{k_1+1}j_{k_1}}a_{j_{k_1}j}$$

for all $j=1,\ldots,n$. The elements of the permutation τ are indices of each monomial. The right-ordered cycle notation of the permutation τ is written as follows:

$$\tau=\left(j_{k_r+l_r}\ldots j_{k_r+1}j_{k_r}\right)\left(j_{k_2+l_2}\ldots j_{k_2+1}j_{k_2}\right)\ldots\left(j_{k_1+l_1}\ldots j_{k_1+1}j_{k_1}j\right).$$