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Taylor's formula involving generalized fractional derivatives

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ABSTRACT

In this paper, we establish a generalized Taylor expansion of a given function f in the form

$$f(x) = \sum_{j=0}^{m} c_j^{\alpha,\rho} (x^\rho - a^\rho)^{j\alpha} + e_m(x)$$

with $m \in \mathbb{N}_0$, $c_j^{\alpha,\rho} \in \mathbb{R}$, $x > a > 0$ and $0 < \alpha \leq 1$. In case $\rho = \alpha = 1$, this expression coincides with the classical Taylor formula. The coefficients $c_j^{\alpha,\rho}$, $j = 0, \ldots, m$ as well as the residual term $e_m(x)$ are given in terms of the generalized Caputo-type fractional derivatives. Several examples and applications of these results for the approximation of functions and for solving some fractional differential equations in series form are given in illustration.

1. Introduction

Fractional calculus has intensively developed since its introduction in the seventies and is nowadays a vividly growing research field [3,19,22,39,40]. The basic idea behind the fractional calculus is to extend to real or complex orders the classical integrals and derivatives of integer orders. Consequently, it provides a useful and powerful tool to solve differential and integral equations as well as various mathematical and physical problems involving nonlocal effects or memory effects, such as quantum mechanics [17,29,37], biophysics [36], fluid mechanics [16], control theory and dynamical systems [18,26], diffusion and wave equations [33], viscoelasticity [6,15], signal processing [23,35], random processes [24,25], and so on. In the literature, several different fractional derivatives have been introduced, e.g. Riemann–Liouville, Caputo, Hadamard, Erdélyi–Kober, Hadamard, Grünwald–Letnikov, Marchaud and Riesz among others [9,10,21,28,31,32].

Recently the author in [12] has introduced a new fractional integral which generalizes into a single form the Riemann–Liouville and the Hadamard integrals. Later on, he has shown that the generalized fractional integral operator is invertible and he has introduced in [13] a new fractional derivative, which generalizes the Riemann–Liouville and the Hadamard derivatives. More recently, the authors in [11] have studied the generalized fractional derivative in Caputo sense. Particularly, they have established that the generalized Caputo-type fractional derivative converges toward the Caputo-type Riemann–Liouville and the Caputo-type Hadamard derivatives when a parameter (denoted ρ) goes to zero and one respectively (see [11, Theorem 3.11] or Theorem 2 hereafter). In physical framework, this new family of generalized fractional derivatives might be more flexible than the classical Riemann–Liouville and Hadamard derivatives since it allows more freedom degrees. Indeed, it is shown in [3] that the generalized fractional derivatives applied to fractional chaotic equations can improve security of image encryption results. Another potential application has also been suggested in quantum mechanics [2].

Following a pioneer work due to Trujillo et al. [38] dealing with Taylor's formula with Riemann–Liouville derivatives, and extended later on to the case of Riemann–Liouville derivatives in Caputo sense by Odibat and Shawagfeh [27], we propose

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to investigate in this paper the Taylor formulas involving the generalized Caputo-type fractional derivatives. We shall see that under some assumptions on the functional spaces, most of the classical formula such as the mean value theorem and Taylor's expansions extend in natural way to the case of generalized Caputo-type fractional derivatives.

The paper is organized in the following way. After some definitions and notations in Section 2, we establish in Section 3 the Taylor expansion of a given function by means of its generalized Caputo-type fractional derivatives. In Section 4, we provide an approximation of a given function in terms of Müntz-like polynomials and we give several examples in illustration. Finally, we apply these results to find the solutions of some fractional differential equations (fde) in series form in Section 5. We also provide in Appendices some auxiliary results as well as an iterative algorithm to compute the jth generalized fractional derivative function of the classical derivatives up to the jth order.

2. Definitions and notations

Throughout this paper, $\mathbb{N}_0$ will denote the set of non-negative integers, a and b will denote two given real numbers such that $0 < a < b < \infty$ and ρ a positive real number. Unless otherwise mentioned, α will denote a complex number such that $Re(\alpha) > 0$. We will use the notation $\lfloor x \rfloor$ to design the integer part of a real number x, that is the greatest integer less than or equal to x. We also define $\lfloor x \rceil = \lfloor x \rfloor + 1$ if $x \notin \mathbb{N}_0$ and $\lfloor x \rceil = x$ if $x \in \mathbb{N}_0$. As in [14], we introduce the weighted L^p spaces, denoted $X_c^p(a,b)$ $(c \in \mathbb{R},\ 1 \le p \le \infty)$ which consist of complex valued Lebesgue measurable functions f on $[a, b]$ for which $\|f\|_{X_c^p(a,b)} < \infty$ where

$$\|f\|_{X_c^p(a,b)} := \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{1/p} \quad \text{for} \quad 1 \leqslant p < \infty$$

and

$$\|f\|_{X_c^\infty(a,b)} := \operatorname*{ess\ sup}_{x \in [a,b]} [x^c |f(x)|].$$

The set of absolutely continuous functions on $[a, b]$ will be denoted $AC[a, b]$. Then we define

$$AC_\gamma^n[a,b] := \left\{ f : [a,b] \to \mathbb{C},\ \gamma^{n-1} f \in AC[a,b] \right\}$$

with $\gamma := x^{1-\rho} \dfrac{d}{dx}$ and $AC_\gamma^1 = AC[a, b]$. It has been shown in [11] that the space $AC_\gamma^n[a, b]$ consists of those and only those functions f which are represented in the form

$$f(x) = \sum_{k=0}^{n-1} c_k (x^\rho - a^\rho)^k + \int_a^x (x^\rho - s^\rho)^{n-1} g(s)\, ds$$

with $g \in L^1(a, b)$ and $c_k \in \mathbb{R}$. In Appendix A we prove the following inclusions:

$$C^n[a,b] \subset AC_\gamma^n[a,b] \subset C^{n-1}[a,b] \subset \cdots \subset C^1[a,b] \subset AC_\gamma^1[a,b] \subset C[a,b]$$

where $C^n[a, b]$ is the set of continuously differentiable functions up to order n.

For $f \in X_c^p(a, b)$, the left-sided generalized fractional integral $\mathcal{I}_{a^+}^{\alpha,\rho} f$ of order α is defined for any real number $x > a$ by:

$$\mathcal{I}_{a^+}^{\alpha,\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho-1} f(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}}\, d\tau. \tag{1}$$

This definition is a fractional generalization of the n-fold left integral of the form

$$\mathcal{I}_{a^+}^{n,\rho} f(x) = \int_a^x t_1^{\rho-1}\, dt_1 \int_a^{t_1} t_2^{\rho-1}\, dt_2 \ldots \int_a^{t_{n-1}} t_n^{\rho-1} f(t_n)\, dt_n.$$

Similarly, the right-sided generalized fractional integral $\mathcal{I}_{b^-}^{\alpha,\rho} f$ of order α is defined for any real number $x < b$ by:

$$\mathcal{I}_{b^-}^{\alpha,\rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\rho-1} f(\tau)}{(\tau^\rho - x^\rho)^{1-\alpha}}\, d\tau. \tag{2}$$

The corresponding generalized fractional derivatives to these generalized integrals are given in what follows.

Definition 1. Let $\alpha \in \mathbb{C}$ with $Re(\alpha) \ge 0$ and $\rho > 0$. Let $n = \lfloor Re(\alpha) \rfloor + 1$ and $f \in AC_\gamma^n[a, b]$. The generalized fractional derivatives relative to the generalized integrals (1) and (2) are given for any real number $0 \le a < x < b < \infty$, respectively, by:

$$\begin{aligned} \mathcal{D}_{a^+}^{\alpha,\rho} f(x) &:= \left(x^{1-\rho} \frac{d}{dx} \right)^n \left(\mathcal{I}_{a^+}^{n-\alpha,\rho} f \right)(x) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(x^{1-\rho} \frac{d}{dx} \right)^n \int_a^x \frac{\tau^{\rho-1} f(\tau)}{(x^\rho - \tau^\rho)^{\alpha-n+1}}\, d\tau \end{aligned}$$