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Short Communication

Optimal perturbation bounds for the core inverse

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ABSTRACT

In this short note, we study some perturbation properties of the core inverse. We present the closed form and perturbation bounds for the core inverse under some conditions, which extend the classical result on the perturbation of the nonsingular matrix. Our expressions for the perturbation of the core inverse are simple and perturbation bounds are sharp.

1. Introduction and preliminaries

There are lots of papers on the core inverse and its applications [1–3,5,7,9,11,13–19]. There is a recent monograph [6] on the algebraic properties of the generalized inverse. In this note, $\mathbb{C}^{m\times n}$ is the set of $m\times n$ complex matrices. If $m=n$, then the identity matrix of order n and the null matrix in $\mathbb{C}^{n\times n}$ are denoted by I_n and $\mathbf{O}$, respectively. For $A\in\mathbb{C}^{m\times n}$, we denote $\mathcal{R}(A)$ for its range and $\mathcal{N}(A)$ for its null space. A^* is the conjugate transpose of the matrix A, and $\|\cdot\|$ denotes the spectral norm.

The *Moore–Penrose inverse* of $A\in\mathbb{C}^{m\times n}$ is the unique matrix $A^\dagger\in\mathbb{C}^{n\times m}$ satisfying the following four equations [6]

$$AA^\dagger A=A,\quad A^\dagger AA^\dagger=A^\dagger,\quad (AA^\dagger)^*=AA^\dagger\quad (A^\dagger A)^*=A^\dagger A. \tag{1}$$

The *Drazin inverse* of $A\in\mathbb{C}^{n\times n}$ is the unique matrix $A^D\in\mathbb{C}^{n\times n}$ satisfying the following three equations [6]

$$A^DA=AA^D,\quad A^DAA^D=A^D,\quad A^{l+1}A^D=A^l\quad \text{for all } l\geq k, \tag{2}$$

where k is the smallest nonnegative integer satisfying $\text{rank}(A^{k+1})=\text{rank}(A^k)$, k is called the *Drazin index* of A and is denoted by $\text{ind}(A)$. Clearly, $\text{ind}(A)=0$ if and only if A is nonsingular. If $\text{ind}(A)=1$, then the Drazin inverse is called the *group inverse* of A and denoted by A_g.

According to Hartwig and Spindelböck's decomposition [8], every matrix $A\in\mathbb{C}^{n\times n}$ of rank r can be represented by

$$A=U\begin{pmatrix}\Sigma K & \Sigma L\\ \mathbf{O} & \mathbf{O}\end{pmatrix}U^*, \tag{3}$$

where $U\in\mathbb{C}^{n\times n}$ is unitary and $\Sigma=\text{diag}(\sigma_1 I_{r_1},\sigma_2 I_{r_2},\ldots,\sigma_r I_{r_t})$ is the diagonal matrix of singular values of A, $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_t>0$, $r_1+r_2+\cdots+r_t=r$, and $K\in\mathbb{C}^{r\times r}$ and $L\in\mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^*+LL^*=I_r. \tag{4}$$

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It follows from (3) that the Moore–Penrose inverse of A is

$$A^{\dagger} = U\begin{pmatrix} K^*\Sigma^{-1} & \mathbf{0} \\ L^*\Sigma^{-1} & \mathbf{0} \end{pmatrix}U^*.$$

If $\text{ind}(A) \le 1$ and K is invertible, then the group inverse of A is

$$A_g = U\begin{pmatrix} K^{-1}\Sigma^{-1} & K^{-1}\Sigma^{-1}K^{-1}L \\ \mathbf{0} & \mathbf{0} \end{pmatrix}U^*.$$

An important role is played by the core inverse, which is given by the following definition.

Definition 1.1. [1] Let $\in \mathbb{C}^{n\times n}$ and $\text{ind}(A) \le 1$. A matrix $A^{\#} \in \mathbb{C}^{n\times n}$ satisfying

$$AA^{\#} = P_A, \qquad \mathcal{R}(A^{\#}) \subseteq \mathcal{R}(A) \tag{5}$$

is called the core inverse. P_A is an orthogonal projection onto $\mathcal{R}(A)$.

Wang and Liu [19, Theorem 2.1] present another characterization of the core inverse.

Definition 1.2. Let $A \in \mathbb{C}^{n\times n}$ and $\text{ind}(A) \le 1$. Then the core inverse of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations

$$AXA = A, \quad AX^2 = X, \quad (AX)^* = AX. \tag{6}$$

We recall some properties of the core inverse in the following result.

Lemma 1.1. *[1, 8] Let $A \in \mathbb{C}^{n\times n}$ be of the form (3) and $\text{ind}(A) \le 1$. Then*

$$A^{\#} = U\begin{pmatrix} (\Sigma K)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}U^*, \tag{7}$$

and

$$AA^{\#} = U\begin{pmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}U^* = AA^{\dagger}, \quad A^{\#}A = U\begin{pmatrix} I_r & K^{-1}L \\ \mathbf{0} & \mathbf{0} \end{pmatrix}U^* = A_gA. \tag{8}$$

It follows from [21] that $\|I - A^{\#}A\| = \|I - A_gA\| = \|A_gA\| = \|A^{\#}A\|$.

If the matrix A is nonsingular and the perturbation E satisfies $\|A^{-1}E\| < 1$, then the inverse of $B = A + E$ exists and we obtain [6]

$$B^{-1} = (I + A^{-1}E)^{-1}A^{-1} = A^{-1}(I + EA^{-1})^{-1}, \tag{9}$$

and

$$\frac{\|A^{-1}\|}{1 + \|A^{-1}E\|} \le \|B^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}E\|}. \tag{10}$$

Our main contribution of this note is to investigate some perturbation properties of the core inverse. We extend the classical results (Eqs. (9) and (10)) to the core inverse under reasonable conditions. Our expressions for the perturbation of the core inverse are simple and perturbation bounds are sharp.

The short note is organized as follows. In Section 2, we give the closed form with optimal perturbation bounds for the core inverse under two-sided perturbations, which is similar to the nonsingular matrix in Eqs. (9) and (10). In Section 3, we present another perturbation for the core inverse with weaker assumption. A concluding remark is given in Section 4.

2. Two-sided perturbation bounds

In this section, we present the closed form and optimal perturbation bounds for the core inverse like the invertible matrix in Eqs. (9) and (10) under two-sided perturbations.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be of the form (3) and $\text{ind}(A) \le 1$, $B = A + E \in \mathbb{C}^{n\times n}$. If the perturbation E satisfies $AA^{\#}E = EAA^{\#} = E$ and $\|A^{\#}E\| < 1$, then*

$$B^{\#} = (I + A^{\#}E)^{-1}A^{\#} = A^{\#}(I + EA^{\#})^{-1}, \tag{11}$$

and

$$BB^{\#} = AA^{\#}, \quad B^{\#}B = A^{\#}A + (I + A^{\#}E)^{-1}A^{\#}E(I - A^{\#}A). \tag{12}$$