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A boundary value problem arising from nonlinear viscoelasticity: Mathematical analysis and numerical simulations

R. Cipolatti [a,*], I.-S. Liu [a], L.A. Palermo [b], M.A. Rincon [a], R.M.S. Rosa [a]

[a] Instituto de Matemática, Universidade Federal do Rio de Janeiro, Brazil
[b] CENPES, Petrobras, Rio de Janeiro, Brazil

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ABSTRACT

The large deformations of solid structures are necessarily described by nonlinear constitutive equations and its effective calculation leads to nonlinear boundary value problems. In this article we apply the successive linear approximation method by considering the relative Lagrangian formulation to describe the deformations of a nearly incompressible viscoelastic material. We prove the existence, uniqueness and regularity of weak solutions for the boundary value problem associated with each step of the method and we perform numerical simulations for the problem of borehole closing in well drilling.

1. Introduction

In the study of deformations of geological structures, it is crucial to consider more realistic constitutive equations for the materials under analysis, which are in general described by nonlinear laws. This is the situation, for instance, in the study of salt dynamics, put in evidence by the oil and gas industry, since the large deformations and the enormous internal tensions deserve special attention.

Concerning the geological phenomena of interest for the petroleum exploration, we can mention, among others, the following ones, which are related to sediment-salt migration, such as the formation of salt diapirs, multiple salt domes and borehole closing, involving very large deformation and creep motions, typical behavior of viscoelastic material bodies. These problems have been widely studied in petroleum industry, where most results for salt tectonics are modeled by regarding the bodies as viscous fluids [11] instead of solid bodies to avoid the numerical difficulties due to large deformation and nonlinearity. A critical situation is the one related to borehole closing, which deserves special attention in the engineering processes.

In this paper we consider a compressible isotropic viscoelastic body $\mathcal{B}$ whose constitutive law for the stress tensor T is given by

$$T = \mathcal{F}(F, \dot{F}) := -pI + s_1 B + s_2 B^{-1} + 2\mu_1 D + \mu_2 (DB + BD) + \mu_3 (DB^{-1} + B^{-1} D), \tag{1.1}$$

* Corresponding author.
E-mail address: cipolatti@im.ufrj.br (R. Cipolatti).

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which is proposed in [10] to describe the salt migration, and which exhibit the main characteristics of salt diapirism. Following standard notation, p is the pressure, I denotes the three-dimensional identity tensor, F is the deformation gradient tensor, $B = FF^{\mathrm{T}}$ is the left Cauchy–Green tensor, D is the symmetric part of the velocity gradient $L = \dot{F}F^{-1}$, i.e.,

$$D := \frac{1}{2}(L + L^{\mathrm{T}}) = \frac{1}{2}\big(\nabla v + (\nabla v)^{\mathrm{T}}\big) \tag{1.2}$$

and s_1, s_2, μ_1, μ_2, μ_3 are material parameters.

If κ denotes a reference configuration of the body $\mathcal{B}$, the equation of motion at time τ can be written in the Lagrangian formulation as

$$\rho_\kappa(X)\ddot{x}(\tau, X) - \mathrm{Div} T_\kappa(\tau, X) = \rho_\kappa(X) g(\tau, X), \quad X \in \kappa(\mathcal{B}), \tag{1.3}$$

where Div denotes de divergent with respect to the space variable X, $x = \chi(\tau, X)$ is the deformation, ρ_κ is the density, T_κ is the associated Piola–Kirchhoff stress tensor at time τ relative to the reference configuration κ and g is the body forces.

Since linearization is physically inadmissible for large deformations, any initial-boundary value problem with (1.3) involves necessarily nonlinear systems of partial differential equations. In order to circumvent the difficulty due to the nonlinearities, we propose an algorithm to solve numerically these boundary value problems based on *successive linear approximations* (SLA), by considering the *relative Lagrangian formulation*. Roughly speaking, it can be regarded as a kind of "time discretization of the Eulerian formulation", where the constitutive equations are calculated at each state, regarded as the reference configuration for the next state. So, by considering sufficiently small time intervals, we can linearize the constitutive functions and solve the corresponding linear boundary value problem between each state. This procedure is similar to the theory of small deformation superposed on finite deformations (see [8]).

Using this algorithm, we have implemented numerical simulations for elastic Mooney–Rivlin materials in particular cases where the exact solutions are known, as in pure shear [9] and in bending of rectangular block into a circular one [8]. The comparison of the numerical results with the known exact solutions of these two examples has confirmed the efficiency of the method in the simulation of large deformations (see also [10] for the simulation of salt diapirism where this approach has also been applied).

The main purpose of the present work is the mathematical analysis of the elliptic boundary value problem coming from the linearization and discretization of (1.3) for a nearly incompressible material in the quasi-static regime, which is effectively solved in numerical simulations.

We organize this paper as follows: in Section 2 we present a brief description of the successive linear approximation method based on the relative Lagrangian formulation. In Section 3 we introduce a family of (linear) boundary problem issued from (1.1), each one corresponding to the problem to be solved numerically in each step of the relative Lagrangian formulation. The mathematical analysis of these linear problems is presented in Section 4 and finely, some numerical simulations are reported in Section 5.

2. Relative Lagrangian formulation and successive linear approximation

Let κ_0 be a fixed reference configuration of a material body $\mathcal{B}$ at time t_0 and κ_t be its deformed configuration at the present time t. It is well known that a certain quantity β related to the motion of $\mathcal{B}$ can be described in Lagrangian formulation or in Eulerian formulation. In the first case β must be described by a function with domain $\mathbb{R} \times \kappa_0(\mathcal{B})$, while in the last case, β must be defined on $\{t\} \times \kappa_t(\mathcal{B})$. In the relative Lagrangian formulation, we regard the magnitude β as a function with domain $\mathbb{R} \times \kappa_t(\mathcal{B})$.

To be more precise, let $x = \chi(t, X)$, $X \in \kappa_0(\mathcal{B})$, be the deformation of $\kappa_0(\mathcal{B})$ to $\kappa_t(\mathcal{B})$ and, for some time $\tau > t$, denote κ_τ the configuration of $\mathcal{B}$ at time τ. Then, for $x = \chi(t, X)$, we write

$$\chi_t(\tau, x) := \chi(\tau, X). \tag{2.1}$$

The function $\chi_t : \mathbb{R} \times \kappa_t(\mathcal{B}) \to \mathbb{R}^3$ defined by (2.1) is the *relative deformation* at time τ with respect to the present time t.

If $F(t, X) = \nabla_X \chi(t, X)$ denotes the deformation gradient, we also consider

$$F_t(\tau, x) := \nabla_x \chi_t(\tau, x),$$

which is the *relative deformation gradient*. Hence, one may check by applying formally the chain rule that

$$F_t(\tau, x) = F(\tau, X) F(t, X)^{-1}.$$

With these notations, we define the *relative displacement* u_t by

$$u_t(\tau, x) := \chi(\tau, x) - x, \quad \forall x \in \kappa_t(\mathcal{B})$$

and the *relative displacement gradient* at time τ as

$$H_t(\tau, x) := \nabla_x u_t(\tau, x) = F_t(\tau, x) - I = F(\tau, X) F(t, X)^{-1} - I. \tag{2.2}$$

These definitions lead to the following relations:

$$F_t(\tau, x) = I + H_t(\tau, x) \quad \text{and} \quad F(\tau, X) = \big(I + H_t(\tau, x)\big) F(t, X). \tag{2.3}$$