Numerical differentiation by a Fourier extension method with super-order regularization

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ABSTRACT

Based on the idea of Fourier extension, we develop a new method for numerical differentiation. The Tikhonov regularization method with a super-order penalty term is presented to deal with the illposdness of the problem and the regularization parameter can be chosen by a discrepancy principle. For various smooth conditions, the solution process of the new method is uniform and order optimal error bounds can be obtained. Numerical experiments are also presented to illustrate the effectiveness of the proposed method.

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1. Introduction

Numerical differentiation is an interesting topic in the field of numerical analysis. It arises in a lot of mathematical models and engineering problems, for instance, solutions related to some Volterra equation [1]; in the processes of image processing [2,3] and identification [4]; some inverse problems arising from financial mathematics [5,6], etc. The main difficulty of numerical differentiation is that it is an ill posed problem, i.e., arbitrarily small error in the input data may cause huge errors in its approximate derivatives. In the past years, a wide range of computational methods has been reported to treat the numerical differentiation problem [7–20]. According to the type of regularization techniques, these methods can be classified into difference methods, mollification methods, truncation method and Tikhonov methods.

In [20], a truncated Fourier series method has been proposed for numerical differentiation. This method is effective for calculating arbitrary derivatives of periodic functions. The theoretical analysis shows that the smoother the original function, the higher the convergence rate of its approximate derivatives. Moreover, the truncated parameter is uniform for the different order derivatives and convergence rates are self-adaptive. However, the situation changes completely when function is nonperiodic. The reason lies in the well-known Gibbs phenomenon, rapidly convergent of Fourier series is available only for periodic functions.

Recently, Fourier extension method has attracted more and more attention of researcher [21–26]. It has been proved successfully to ameliorate the Gibbs phenomenon. For a function $f \in H^p(-1, 1)$, the idea of the Fourier extension is to extend the function f to a function g that is periodic on a interval $[-T, T]$ with $T > 1$. (In this paper, we focus on $T = 2$.) In Fig. 1 the function $f(x) = \exp(x)$, a periodic Fourier extension obtained by the method of [23], has been shown. For numerical differentiation, our aim is to obtain approximation derivatives of f from its perturbed data f^δ satisfying

$$\|f - f^\delta\|_{L^2(-1,1)} \leq \delta, \tag{1}$$

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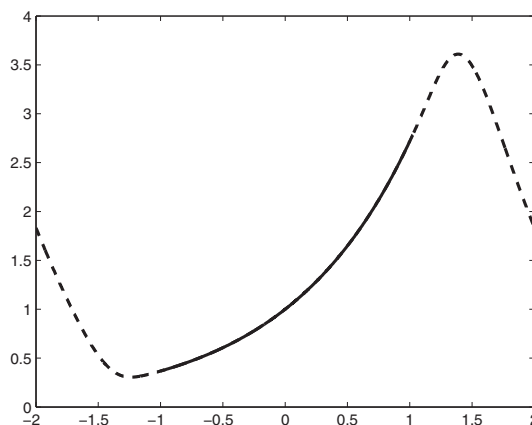*E-mail address:* wozitianshanglai@163.com (Z. Zhao).

Fig. 1. The function $f(x) = \exp(x)$ (solid) on the interval $[-1, 1]$ and its periodic extension approximation (dashed–dotted) on the interval $[-2, 2]$.

where $\delta > 0$ is a given constant called the error level. First, we construct an approximation function $\mathfrak{f}$ by using Fourier extension method and $\mathfrak{f}^{(k)}$ will be used as the approximation of $f^{(k)}$. Different from the SVD (Singular Value Decomposition) method which is used in previous literatures, we present a modified Tikhonov regularization method to obtain a stable Fourier extension and a new penalty term will be used in the modified method. From the theories results of regularization in Hilbert scales [27], if we use $\|D^q \mathfrak{f}\|^2$ (D is one order differential operator) as the penalty term in Tikhonov functional, then we can obtain $\|D^k(\mathfrak{f} - f)\| = O(\delta^{\frac{p-k}{p}})$ whether $q \geq p$ (high order regularization) or $p/2 < q < p$ (low order regularization). But when p is large, above theoretical result is difficult to attain in practice subject to the limitation of floating point precision. In this paper, we use $\|\sum_{l=0}^{\infty} \frac{D^l}{l!} \mathfrak{f}\|^2$ as the penalty term (named it supper order regularization) and the theoretical result shows that $\|D^k(\mathfrak{f} - f)\| = O(\delta^{\frac{p-k}{p}})$ holds for any $p \in \mathbb{R}$ when we choose the regularization parameter by a discrepancy principle.

The structure of the paper is as follows. In Section 2, we present the mathematical formulation of the Fourier extension by using Tikhonov method with supper order regularization. The choice of the regularization parameter and corresponding convergence results are shown in Section 3. Section 4 provides a variety of numerical results to demonstrate the effectiveness of our method.

2. Formulation of problem and solution

We first introduce some notation. Let $\Lambda_1 = (-1, 1)$, $\Lambda_2 = (-2, 2)$. For any $\mathbf{v} = (c_0, c_1, s_1, \ldots, c_n, s_n, \ldots)^T \in l^2$, we define the operator

$$(\mathcal{F}\mathbf{v})(x) = \frac{c_0}{2} + \sum_{l=1}^{\infty} \left(c_l \cos \frac{\pi}{2} lx + s_l \sin \frac{\pi}{2} lx \right). \tag{2}$$

and

$$\mathcal{D}\mathbf{v} = \left(0, \frac{\pi}{2} c_1, \frac{\pi}{2} s_1, \ldots \frac{n\pi}{2} c_n, \frac{n\pi}{2} s_n, \ldots \right)^T,$$

$$\mathcal{R}\mathbf{v} = \left(\sum_{k=0}^{\infty} \frac{\mathcal{D}^k}{k!} \right) \mathbf{v} = (c_0, e^{\frac{\pi}{2}} c_1, e^{\frac{\pi}{2}} s_1, \ldots e^{\frac{n\pi}{2}} c_n, e^{\frac{n\pi}{2}} s_n, \ldots)^T, \tag{3}$$

$$\mathcal{P}_N \mathbf{v} = (c_0, c_1, s_1, \ldots c_N, s_N, 0, 0, \ldots)^T.$$

Remark 1. For any $f \in H^k(\Lambda_2)$, if the vector $\mathbf{v}_f$ satisfy $(\mathcal{F}\mathbf{v})(x) = f(x)$. Then it can be derived that

$$\|D^k f\|_{L^2(\Lambda_2)} = \|\mathcal{D}^k \mathbf{v}_f\|_{l^2}. \tag{4}$$

Now we define the following cost functional:

$$\Phi(\mathbf{v}) = \|\mathcal{F}\mathbf{v} - f^\delta\|^2_{L^2(-1,1)} + \alpha \|\mathcal{R}\mathbf{v}\|^2_{l^2}, \tag{5}$$

where α is a regularization parameter. Now if let $\mathbf{v}_\alpha^\delta$ is the minimizer of the functional Φ, then $\mathfrak{f}^\delta := \mathfrak{f}_\alpha^\delta = \mathcal{F}\mathbf{v}_\alpha^\delta$ will be chosen as the approximate function of f. It is well known that $\mathbf{v}_\alpha^\delta$ can be obtained by solving the following equation [28]:

$$\left(\mathcal{F}^*\mathcal{F} + \alpha \mathcal{R}^2 \right)\mathbf{v} = \mathcal{F}^* f^\delta. \tag{6}$$