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Heterogeneity and chaos in congestion games

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ABSTRACT

We analyze a class of congestion games where agents use resources to send a finite amount of goods from an initial location to a terminal one. The resources are costly and costs are load dependent. In this context we concentrate on the heterogeneity because we not only assume that agents have limited computational capability but also they differ in the quantities they must send and the reactivity. We introduce an appropriate dynamical system, which has the steady state exactly at the unique Nash equilibrium of the static congestion game, and we investigate the dynamical behavior of the game. We provide a closed form characterization on the unique Nash equilibrium in the underlying static congestion game and we prove that the equilibrium crucially depends on the aggregate congestion. Not only do we provide a local stability condition in terms of the agents' reactivity and the nonlinearity of the cost functions but also we study the role of the heterogeneity in this context. We show analytically that heterogeneity can be destabilizing with a cascade of flip-bifurcation leading to periodic cycles and finally to chaos but also that mastering the degree of heterogeneity can be used to tame and control complex dynamics; moreover in relation to the reactivity levels their product must roughly small. However if both reactivity levels are near but outside the stability region then it is sufficient to act on just one to restore stability. Finally, if both levels are far form the stability region acting on one will not restore stability.

1. Introduction

Congestions and strategic behavior are important parts of our economic world. Many important economic realities can be understood by analyzing the strategic behavior of agents and the congestion determined by the agents. It is fair to say that many important challenges in the future depend on the way we will be able to mitigate the effects of these two important ingredient. For this reason there is a constant and large interest in the study of congestion games (for basic properties see [15,23] and also [22] for an interesting example). In fact this type of games have enough structure to model very interesting strategic situations encompassing oligopoly and the internet (see [26]).

In a previous paper (see [16]) we studied, in a very simple congestion game, the dynamics arising when we assume that our players were adopting a gradient rule to react to other people's decisions and we introduced an important parameter in order to control the level of reactivity of our agents. By all means, we are not the first to study such a mechanism since they

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were extensively studied in the literature in the context of oligopolist competition and rent-seeking games as the reader can see by reading [4–8,30]. Moreover, such a mechanism has been investigated in relationship to the role of heterogeneity in the named games as the reader can find in [1,2,9,10,29] and also in other contexts as the prisoner's dilemma and public good games by Perc and co-authors (see [3,13,18–21]).

In this paper we depart from one of our main assumptions: the homogeneity of the players. In fact we posit that our agents are not be identical and we introduce heterogenous players. The rationale to do so is that it is quite important to understand if there is the possibility for the authority to intervene in the market in order to correct its failures. In a congestion game one of the most striking failures of the market is the creation of chaotic behavior. We show that the presence of heterogenous agents give us the freedom to introduce a player that can mitigate the chaos and for this reason we like to think of one of our agents as a social's planner agent which is in the game just to make sure that chaos can be tamed. We find this quite appealing especially if we think of situations such as traffic in cities and the provision of public goods such as buses (see also [14]). It is quite important also to stress that the level of knowledge required for such interventions is not too high and therefore, at least from the qualitative point of view, realistic.

The paper is organized as follows. In the second section we present the relevant definitions of congestion games and we characterize the Nash equilibrium. Hence, in the third section, we study the dynamic setting and we explicitly find analytic conditions to characterize stability and its relation to the agents' reactivity parameters and heterogeneity. In the fourth section several numerical simulations are presented in order to illustrate our results. In the last section we point out the conclusions and we outline possible extensions.

2. Congestion games with heterogeneity

In this section we very briefly present the most important definitions and we prove the main result of the static congestion game. Congestion games are an important class of games first introduced by Rosenthal in 1973 (see [15],[23]). In a congestion game there are players and resources, and the payoff of each player depends on the resources the agent chooses and the number of players choosing the same resource. Congestion games are a special case of potential games. Rosenthal proved that any congestion game is a potential game and Monderer and Shapley [15] proved that for any potential game there exists a congestion game with the same potential function. In this paper we consider a network where there are two players and there are two nodes and the players need to send goods from the initial node to a final node. We assume that the initial node is connected to the terminal node via connection edges which we think as resources. However, both connection edges get easily congested, meaning the more goods pass through an edge the greater the cost for each player becomes; so having both players go through the same connection edge causes extra costs to both players. Of course the desirable outcome in this game would be for the two players to "coordinate" and send goods through different edges since they are acting strategically. But the problem is what outcome will rise at equilibrium and what will the cost be for each player at equilibrium. To investigate this aspect we need to be more formal so we are going to present the game in a mathematically tractable way as is done in Game Theory.

There are two nodes, the source and the target, that we call S and T and there are two possible paths that connect them, we call the paths e and f and we refer to them as resources. The set of resources is $\mathcal{E} = \{e, f\}$ which must be shared among two players and each resource $a \in \mathcal{E}$ has a load-dependent cost given by $c_a : \mathbb{R}_+ \to \mathbb{R}_+$. We denote with $\mathcal{L}$ the set of all cost functions and we assume that the costs functions are of the form $c_a(x) = x \cdot \ell_a(x)$ where ℓ_a is a function. It is worth to stress that the cost faced by players depends only on the chosen path and the amount of flow on the edge but not on the identity of the player. Each player $i = 1, 2$ must send d_i units from the node S to the node T, we can summarize the choice by giving the pairs (x_e^i, x_f^i) with

$$x_e^i + x_f^i = d_i$$

Given a strategy profile $x = (x^i)_{i=1,2}$ and a resource $a \in \mathcal{E}$ we call

$$x_a^1 + x_a^2 \overset{def}{=} x_a \qquad \text{the total load on } a.$$

Given a strategy profile $x = (x^i)_{i=1,2}$ the cost for player $i = 1, 2$ is defined as

$$c_i(x) = \sum_{a \in \mathcal{E}} x_a^i \cdot \ell_a(x_a)$$

and each player tries to maximize $-c_i(x)$. The social cost is, by definition,

$$c(x) = \sum_{i=1}^{2} c_i(x) = \sum_{i=1}^{2} \sum_{a \in \mathcal{E}} x_a^i \cdot \ell_a(x_a).$$

Of course the most important aspects of this game are the equilibria of the game. A pure strategy Nash equilibrium is a strategy profile $x = (x^i)_{i=1,2}$ such that for each agent the following:

$$c_i(x) \leq c_i(y^i, x^{-i}) \qquad \text{for every } y^i \in S_i$$