Applied Mathematics and Computation 335 (2018) 305–312

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An approximation scheme for the time fractional convection–diffusion equation☆

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ARTICLE INFO

Keywords:
Time fractional convection–diffusion equation
Caputo derivative
Stability
Convergence
Finite difference method

ABSTRACT

In this paper, a discrete form is proposed for solving time fractional convection–diffusion equation. Firstly, we obtain a time discrete scheme based on finite difference method. Secondly, we prove that the time discrete scheme is unconditionally stable, and the numerical solution converges to the exact one with order $O(\tau^{2-\alpha})$, where τ is the time step size. Finally, two numerical examples are proposed respectively, to verify the order of convergence.

1. Introduction

Many natural phenomena are modeled by partial differential equations, even medical problems [1,2] can also be modeled by partial differential equations. As a class of partial differential equations, integer order convection–diffusion equations are also have a wide range of applications. The growing number of applications of fractional derivatives in various fields [3–6] such that fractional convection–diffusion equations have received an increasing attention in recent years and have been used to model a wide range of problems in sound, heat, electrostatics, electrodynamics, fluid flow, and elasticity.

One of the key issues with fractional convection–diffusion equations is the design of efficient numerical schemes and computational efficiency is an important parameter for the numerical schemes. A variety of numerical methods for the fractional partial differential equations have been proposed, such as finite element methods [7–12], finite difference methods [13–17], spectral methods [18–20], meshless methods [21,22]. In [13], fractional advection–diffusion equation with space–time fractional derivative has been considered and the stability and convergence of the mentioned method has also been discussed. Zhang et al. [14] have proposed a new numerical method for the time mobile–immobile advection–dispersion problem with variable fractional derivative. Zhuang et al. [15] have developed some numerical methods for the nonlinear fractional advection–diffusion equation with variable-order fractional derivative. The authors of [23] have proposed an implicit meshless method based on the radial basis functions for the numerical simulation of time fractional diffusion equation.

In this paper, we aim to propose a time discrete form combined with finite difference methods for the time fractional convection–diffusion equation. We consider the time fractional convection–diffusion equation of the form:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \Delta u(x,t) - \nabla u(x,t) + f(x,t), & (x,t) \in \Omega \times [0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t)\mid_{\partial\Omega} = 0, & t \in [0,T], \end{cases} \tag{1}$$

☆ This work is supported by the NSFC (No. 11461072), the Youth Science and Technology Education Project of Xinjiang (No. QN2016JQ0367) and the Science and Technology Innovation Project for Master Graduate Student of Xinjiang Normal University (No. XSY201602011).

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https://doi.org/10.1016/j.amc.2018.04.019
0096-3003/

where Ω denotes a bounded domain in R^n, $0<\alpha<1$, f and $u_0(x)\equiv u^0$ are given smooth functions, $\Delta u(x,t)=\frac{\partial^2 u(x,t)}{\partial x_1^2}+\cdots+\frac{\partial^2 u(x,t)}{\partial x_n^2}$ and $\nabla u(x,t)=\frac{\partial u(x,t)}{\partial x_1}+\cdots+\frac{\partial u(x,t)}{\partial x_n}$. The fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is Caputo fractional derivative defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}.$$

2. Time discrete scheme

We will introduce a finite difference approximation to discrete the time fractional derivative. Let $t_m:=m\tau$, where $\tau:=T/M$ is the time step, $m=0,1,\ldots,M$ be mesh points, where M is a positive integer.

The time fractional derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ at t_{m+1} is estimated by

$$\begin{aligned}\frac{\partial^\alpha u(x,t_{m+1})}{\partial t^\alpha}&=\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t_{m+1}-s)^\alpha}\\&=\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\frac{u(x,t_{k+1})-u(x,t_k)}{\tau}\int_{t_k}^{t_{k+1}}\frac{ds}{(t_{m+1}-s)^\alpha}+R_\tau^{(1)}.\end{aligned}\tag{2}$$

Using the analysis in [18] the truncation error $R_\tau^{(1)}$ has the following form

$$|R_\tau^{(1)}|\le C_0\left|\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\frac{2s-t_{k+1}-t_k}{(t_{m+1}-s)^\alpha}ds+O(\tau^2)\right|\le C\tau^{2-\alpha},$$

where C_0 and C are constants. Also Eq. (2) can be rewritten as

$$\begin{aligned}\frac{\partial^\alpha u(x,t_{m+1})}{\partial t^\alpha}&=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{m}b_{m-k}(u(x,t_{k+1})-u(x,t_k))+R_\tau^{(1)}\\&=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=0}^{m}b_k(u(x,t_{m-k+1})-u(x,t_{m-k}))+R_\tau^{(1)},\end{aligned}\tag{3}$$

where $b_k=(k+1)^{1-\alpha}-k^{1-\alpha}$, $k=0,1,\ldots,M$. Let $\lambda=\Gamma(2-\alpha)\tau^\alpha$, substituting Eq. (3) into Eq. (1), we obtain the following form

$$\begin{aligned}&u(x,t_{m+1})-\lambda\Delta u(x,t_{m+1})+\lambda\nabla u(x,t_{m+1})\\&=u(x,t_m)-\sum_{k=1}^{m}b_k(u(x,t_{m-k+1})-u(x,t_{m-k}))+\lambda f(x,t_{m+1})+R_\tau^{(2)},\end{aligned}\tag{4}$$

where

$$|R_\tau^{(2)}|\le C^*\tau^2,\tag{5}$$

and C^* is a constant.

Let u^m be the numerical approximation to $u(x,\,t_m)$ and $f^{m+1}=f(x,t_{m+1})$. Now by omitting the small term $R_\tau^{(2)}$ in (4), we can construct the following discrete scheme for solving Eq. (1).

$$\begin{aligned}&u^{m+1}-\lambda\Delta u^{m+1}+\lambda\nabla u^{m+1}=u^m-\sum_{k=1}^{m}b_k(u^{m-k+1}-u^{m-k})+\lambda f^{m+1},\ m=0,1,\ldots,M,\\&u^0=u_0(x),\qquad x\in\Omega\subset R^n,\\&u^m\ \ |_{\partial\Omega}=0,\qquad t\in[0,T].\end{aligned}\tag{6}$$

3. Stability analysis and error estimates

In this section, we discuss the stability and convergence of Eq. (6) using the following lemma. Firstly, we define the functional spaces endowed with standard norms and inner products. The space $L^2(\Omega)$ is equipped with the usual L^2-scalar product

$$(u,v)=\int_\Omega uv\,d\Omega,\tag{7}$$

which induces the L^2-norm

$$\|u\|_2=(u,u)^{1/2},\tag{8}$$

for $u,v\in L^2(\Omega)$.