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Some higher-order iteration functions for solving nonlinear models

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ABSTRACT

In this paper, we present a new efficient sixth-order family of Jarratt type methods for solving scalar equations. Then, we extend this family to the multidimensional case preserving the same order of convergence. We also discuss the theoretical convergence properties of the proposed scheme in the case of scalar as well as multidimensional case. The derivation of these schemes are based on weight function approach and free disposable parameters. We also demonstrate the applicability of them on total six number of problems: first five are real life problems namely, continuous stirred tank reactor (CSTR), chemical engineering, the trajectory of an electron in the air gap between two parallel plates, Hammerstein integration and boundary value problems; last one is the standard academic test problem. In addition, numerical comparisons are made to show the performance of the proposed iterative techniques with the existing techniques of the same order in the scalar as well as multi-dimensional case. Finally on the basis of numerical results, we conclude that our techniques perform better in terms of residual error, error between the two consecutive iterations, asymptotic error constant term and approximated root as compared to the existing ones of same order in scalar as well as multidimensional case.

1. Introduction

In this paper, we consider

$$F(x) = 0, \tag{1.1}$$

where $F : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a univariate function when $n = 1$ or multivariate function when $n > 1$ on an open domain $\mathbb{D}$.

Construction of higher-order iterative methods in order to approximate the solutions of Eq. (1.1) is one of the most important and challenging task in the field of numerical analysis. The importance of this subject led to the development of many numerical techniques. However, most of them are iterative in nature because analytic methods to obtain the exact solution of such problems are almost non existent. Therefore, scholars from the worldwide are trying their best to resort an iterative method from the past few decades. In addition, iterative methods provide an approximated solution corrected up to a specified degree of accuracy. This accuracy further depends upon the considered iterative method and programming software namely, Fortran, Maple, Matlab, Mathematica, etc.

There are several examples where we can see the applicability of them to the real world problems. For example, Moré [20] proposed a collection of nonlinear model problems and most of them are phrased in the terms of $F(x) = 0$. On the other

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hand, Grosan and Abraham [16], also discussed the applicability of the system of nonlinear equations in neurophysiology, kinematics syntheses, chemical equilibrium, combustion and economics modeling problems. In addition, the reactor and steering problems were solved in [2,28] by phrasing them in the form of nonlinear system $F(x) = 0$. Moreover, Lin et al. [19] also discussed the applicability of the system of nonlinear equations in transport theory.

In the past and recent years, researchers proposed a plethora of one point and multi-point methods for obtaining the solutions of the nonlinear equations [1–12,14–34]. Generally, most of them are the extension of the classical Newton method or Newton like method at the expense of some additional functional evaluations of the involved function.

In general, there are two ways to construct new iterative methods for system of nonlinear equations: first one is develop a new scheme for scalar equation then extend the same scheme to the multidimensional case preserving the same order of convergence; second one is based on some other approaches like quadrature formulae, Adomian polynomial, divided difference approach, etc. for constructing iterative schemes for nonlinear systems (the details of such approaches can be seen in some standard text books [5,6,21,22,27]). Some scholars like Cordero et al. [9], Abad et al. [1], Cordero et al. [10] and Wang et al. [30], have use the first type of technique in order to construct iterative methods for nonlinear system. On the other hand, Sharma et al. [26] proposed fourth and sixth-order iterative methods based on weighted-Newton iteration. In the recent years, Artidiello et al. [7] proposed fourth-order methods based on the weight function approach.

It seems that first technique is one of the simplest way to construct new scheme for nonlinear system. But, it is not always possible at least in the same form. In the case of scalar equation, the consider function and its derivative have the same computational cost. But, this is not true in the multidimensional case.

In this paper, our principle aim is to propose a new sixth-order family of Jarratt type methods [18] for scalar equations. Then, we extend this family for the multidimensional case, while preserving the same order of convergence. The derivation of the proposed scheme is based on weight functions approach and some disposable parameters. The efficiency of the proposed methods is tested on a concrete variety of real life problems which also confirm the applicability of our proposed scheme in the case of scalar as well as multidimensional case. Finally, we conclude on the basis of obtained numerical results that our methods perform better than the existing ones.

2. Development of the scheme for scalar equations

In this section, we present a new family of sixth-order Jarratt type methods, which is defined as follows:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)}Q(\upsilon), \\ x_{n+1} &= z_n - \left[\frac{a_1}{f'(x_n)} + \frac{a_2}{f'(x_n)+f'(y_n)}\right]f(z_n), \end{aligned} \tag{2.1}$$

where the weight function $Q: \mathbb{C} \to \mathbb{C}$ is an analytic function in a neighborhood of μ with $\upsilon = \frac{f'(x_n)}{f'(x_n)+f'(y_n)}$ and $a_1,\ a_2 \in \mathbb{R}$ are free disposable parameters. Theorem 1 demonstrates that the order of convergence of scheme (2.1) reaches at maximum sixth-order.

Theorem 1. *Let $f: \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ be an analytic function in the region $\mathbb{D}$ enclosing the required simple zero ξ of the considered function $f(x)$. Further, we also assume that an initial guess x_0 is sufficiently close to ξ. Then, the scheme (2.1) reaches at sixth-order convergence when*

$$a_1 = -2,\ a_2 = 6,\ Q(\mu) = 1,\ Q'(\mu) = 3,\ Q''(\mu) = 24 \tag{2.2}$$

and satisfies the following error equation

$$e_{n+1} = -\frac{(14c_2^2 - 3c_3)}{486}\left(c_2^3(Q'''(\mu) - 450) + 162c_3c_2 - 18c_4\right)e_n^6 + O(e_n^7), \tag{2.3}$$

where $\mu = \frac{1}{2}$ and $Q'''(\mu) \in \mathbb{R}$ is a free disposable parameter.

Proof. Let us consider $e_n = x_n - \xi$ be the error in the n^{th} iteration. The Taylor's series expansion of the function $f(x_n)$ and it's first order derivative $f'(x_n)$ around $x = \xi$ with the assumption $f'(\xi) \neq 0$ leads us to

$$f(x_n) = f'(\xi)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\right), \tag{2.4}$$

where $c_k = \frac{f^{(k)}(\xi)}{k!f'(\xi)}$ for $k = 2,\ 3,\ \ldots 6$ and

$$f'(x_n) = f'(\xi)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)\right), \tag{2.5}$$

respectively.