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Optimal system, invariant solutions and evolution of weak discontinuity for isentropic drift flux model

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ABSTRACT

Inequivalent classes of similarity solutions for isentropic drift flux two phase flow model are obtained by using one dimensional optimal classifications. The optimality of this classification is shown in a constructive manner, i.e., by constructing various invariant functions including conditional and numerical invariants. Further, with the help of invariant solution, we discuss the evolution of weak discontinuity which displays the nature of the general solution.

1. Introduction

Often it is very difficult to find complete exact solutions for nonlinear systems of partial differential equations (PDEs) and one has to handle numerically. But, this approach may not always resolve the issue completely when studying the real world physical problems. In fact instead of solely depending on numerical methods, one can construct specific exact solutions known as invariant solutions. These solutions are useful in designing, analyzing and testing numerical schemes. Moreover, they describe the asymptotic behavior or display the structure of singularity of the general solution, if such singularities exist. One of the most elegant methods, to determine invariant solutions of PDEs, is based on the study of invariance with respect to one parameter Lie group of point transformations [1–4]. This method is widely used for deriving invariant solutions [5–8] of nonlinear systems of PDEs with applications in different fields.

In symmetry analysis, the problem of classifying optimal system [9] is very vital to understand the behavior of the solutions for a given PDE. Given a large group that leaves a system of PDEs invariant, one would like to minimize the search for group invariant solutions. This can be done by finding inequivalent branches of solutions, which leads to the concept of optimal systems. The notion of the optimal system was first introduced by Ovsiannikov [4]. This technique was followed by Galas [10] and Ibragimov et al. [11]. But, Olver [3] slightly modified the construction of optimal system by using adjoint representation and the corresponding classification has been done based on Killing form. These days researchers [12,13] are mostly following the modified method introduced by Olver. Further, the inequivalence among the representatives of an optimal system is first shown by Chou and Qu [14,15]. They have demonstrated the proof of inequivalence in a constructive manner, i.e., by introducing different invariant functions including numerical and conditional invariants (see [14]).

Given an inequivalent class of optimal system, one can obtain invariant solutions by solving PDEs with fewer number of independent variables than the given system of PDEs [16]. In particular, a system of PDEs with two independent variables reduces to a system of ordinary differential equations (ODEs) [17]. This reduced system of ODEs can sometimes be solved analytically and hence, solutions for the original system can be found with the help of similarity transformations. The

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invariant solutions obtained are used to study the behavior of weak discontinuities (C^1 discontinuities) for quasilinear hyperbolic systems [18]. These C^1 discontinuities get transformed to shocks after certain time. This behavior of the solution is very crucial for the hyperbolic system of PDEs.

Multiphase flows consisting of a combination of fluids and gases have wide applications, like oil wells, heat exchangers, etc. [19]. Most of the heat exchangers used in industrial purpose involves two phase flow models (gas–liquid or vapor–liquid). They are used in the power and process industry as well as in air conditioning, refrigerating, and food production. Mostly the latent heat of evaporation of a vapor-liquid mixture is used to improve the transport capacity and also to enhance the heat transfer process. Due to the presence of surface tension and other factors two phase flow problems become nonlinear and are quite complex to solve analytically. This could be due to the inherent physical assumptions. Moreover, many times, these systems are hyperbolic in nature and hence have to deal with complicated wave structure. In this context, obtaining an inequivalent class of invariant solutions becomes handy to understand a specific class of solutions of such complex systems. Hence, in this paper we are concerned with the application of Lie group analysis to the drift-flux model. Such a model is widely used in the formulation of the simple equations for two-phase flow problems. This model supports fast computations and accuracy to handle problems in a wide range of real world applications. Within this framework, we propose to derive analytical solutions for isentropic no-slip drift flux model in a multiphase environment [20] using the Lie group analysis.

The layout of this paper is as follows. In Section 2, symmetry group as well as the structure of its corresponding Lie algebra are discussed. In Section 3, one dimensional optimal classification is done and correctness of the optimality is shown in a constructive manner. Similarity reductions and invariant solutions are included in Section 4. Further, in Section 5 evolution of C^1 discontinuity is studied which is followed by conclusions in Section 6.

2. Lie symmetries and structure of its Lie algebra

The isentropic no-slip two phase drift flux model [20] can be written in the following form:

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t}+\rho_1\frac{\partial u}{\partial x}+u\frac{\partial \rho_1}{\partial x}=0,\\
&\frac{\partial \rho_2}{\partial t}+\rho_2\frac{\partial u}{\partial x}+u\frac{\partial \rho_2}{\partial x}=0,\\
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\gamma a^2(\rho_1+\rho_2)^{\gamma-2}\left(\frac{\partial \rho_1}{\partial x}+\frac{\partial \rho_2}{\partial x}\right)=0,\ x\in\mathbb{R},\quad t>0,\quad \gamma\neq 1,
\end{aligned}\tag{1}$$

where ρ_1, ρ_2 are the densities of phase 1 (gas) and phase 2 (liquid), respectively, u is the common velocity and a is the constant which depends on both phases. Eq. (1) forms a quasilinear hyperbolic system of first order PDEs with two independent variables and three dependent variables. Applying straight forward procedure of symmetry analysis outlined in [2,16], we obtain five dimensional Lie algebra L^5 generated by infinitesimal generators as follows:

$$\begin{aligned}
&X_1=\frac{\partial}{\partial x}\ \text{(translation in } x),\\
&X_2=x\frac{\partial}{\partial x}+\frac{2\rho_1}{\gamma-1}\frac{\partial}{\partial \rho_1}+\frac{2\rho_2}{\gamma-1}\frac{\partial}{\partial \rho_2}+u\frac{\partial}{\partial u}\ \text{(dilation in } x,\ \rho_1,\ \rho_2 \text{ and } u),\\
&X_3=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\ \text{(Galilean boost)},\\
&X_4=\frac{\partial}{\partial t}\ \text{(translation in } t),\\
&X_5=t\frac{\partial}{\partial t}-\frac{2\rho_1}{\gamma-1}\frac{\partial}{\partial \rho_1}-\frac{2\rho_2}{\gamma-1}\frac{\partial}{\partial \rho_2}-u\frac{\partial}{\partial u}\ \text{(dilation in } t,\ \rho_1,\ \rho_2 \text{ and } u).
\end{aligned}\tag{2}$$

Solvability of Lie algebra is very essential for the reduction of independent variables in the case of system of PDEs. Hence, we discuss the structure of L^5 (refer Table 1). It is very elucidate that the Lie algebra L^5 has solvable structure and the corresponding chain of subalgebras, generated by respective basis, is as follows:

$$\{X_1\}\subset\{X_1,X_3\}\subset\{X_1,X_3,X_4\}\subset\{X_1,X_2,X_3,X_4\}\subset\{X_1,X_2,X_3,X_4,X_5\}.$$

3. Classification of subalgebras

3.1. Construction of invariant functions

A real function ϕ on the Lie algebra L^5 is called an invariant if $\phi(Ad_g(x))=\phi(x)$ for all x in L^5, where g be an element of Lie group G generated by L^5 and Ad_g denotes the adjoint representation of G on L^5. Here, we construct different invariant functions including numerical and conditional invariants, those are very much effective while proving the inequivalence of subalgebras in one dimensional optimal classification.