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The preconditioned iterative methods with variable parameters for saddle point problem[☆]

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ABSTRACT

In this paper, by transforming the original problem equivalently, we propose a new preconditioned iterative method for solving saddle point problem. We call the new method as PTU (preconditioned transformative Uzawa) method. And we study the convergence of the PTU method under suitable restrictions on the iteration parameters. Moreover, we show the choices of the optimal parameters and the spectrum of the preconditioned matrix deriving from the PTU method. Based on the PTU iterative method, we propose another iterative method – nonlinear inexact PTU method – for solving saddle point problem. We also prove its convergence and study the choices of the optimal parameters. In addition, we present some numerical results to illustrate the behavior of the considered algorithms.

1. Introduction

In this paper, we consider the following saddle point problem:

$$\mathcal{A}z \equiv \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{n\times m}$ is a matrix of full column rank, and $n \geq m$, B^T denotes the transpose of the matrix B, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are two given vectors.

Many scientific and engineering applications may lead to the linear system (1.1), including mixed or hybrid finite element approximations of second-order elliptic problems [4,30], computational fluid dynamics [15,17,18,20], weighted and equality constrained least squares estimation [3], Maxwell system and inverse problems [13,28], and so on.

As the saddle point problem (1.1) is large and sparse, iterative methods become more attractive than direct methods. Many iteration methods have been proposed for solving the linear system (1.1), including Uzawa-type schemes (see [9,16,19,35,36]), iterative projection methods (see [2]), iterative null space methods (see [1,21,33]), splitting methods (see [5–7,14,22,23]). Recently, there is a rapidly increasing literature which is concerned with preconditioned iterative methods for solving (1.1) (see [8,9,11,12,16,27,31,32]). By the technique similar to [29,37,38], we introduce a perconditioner as follows:

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$$P_\tau = \begin{pmatrix} I_n & 0 \\ B^T & -\tau B^T B Q^{-1} \end{pmatrix}, \tag{1.2}$$

where $Q \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix and $\tau \neq 0$ is a given parameter. Then the saddle point problem (1.1) can be equivalent to

$$\begin{pmatrix} A & B \\ B^T A - \tau B^T B Q^{-1} B^T & B^T B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ B^T f - \tau B^T B Q^{-1} g \end{pmatrix}. \tag{1.3}$$

Let

$$\widehat{M}(\omega,\tau) = \begin{pmatrix} \frac{1}{\omega} A & 0 \\ B^T A - \tau B^T B Q^{-1} B^T & B^T B \end{pmatrix}, \quad \widehat{N}(\omega,\tau) = \begin{pmatrix} (\frac{1}{\omega} - 1) A & -B \\ 0 & 0 \end{pmatrix}.$$

Then the coefficient matrix in (1.3) admits the splitting

$$\begin{pmatrix} A & B \\ B^T A - \tau B^T B Q^{-1} B^T & B^T B \end{pmatrix} = \widehat{M}(\omega,\tau) - \widehat{N}(\omega,\tau). \tag{1.4}$$

Based on the equivalent system (1.3) and the splitting (1.4), we present the following new preconditioned iterative method for solving the saddle point problem (1.1):

Algorithm 1.1. Given $(x_0^T, y_0^T)^T \in \mathbb{R}^{n+m}$, the sequence $(x_k^T, y_k^T)^T \in \mathbb{R}^{n+m}$ is defined for $k = 1, 2, \dots$ as follows:

$$\begin{cases} x_{k+1} = x_k + \omega A^{-1}(f - A x_k - B y_k), \\ y_{k+1} = (B^T B)^{-1} B^T (f - A x_{k+1}) + \tau Q^{-1}(B^T x_{k+1} - g). \end{cases} \tag{1.5}$$

where $\omega \neq 0$ and $\tau \neq 0$ are two given parameters.

For convenience, we call the above iterative method as PTU (preconditioned transformative Uzawa) iterative method. The main difficulty in Algorithm 1.1 is to solve the linear system with $B^T B$, because the matrix $B^T B$ might be dense. So in actual computation, we can firstly apply Cholesky decomposition to $B^T B$ and then respectively solve a lower triangular matrix and a upper triangular matrix at each step.

In this paper, we study the convergence of the PTU iterative method (1.5) under suitable restrictions on the iteration parameters. And we show the choices of the optimal parameters and the spectrum of the preconditioned matrix deriving from the PTU method (1.5). Combining the techniques in [24–26] with the PTU iterative method (1.5), we propose another new method – nonlinear inexact PTU iterative method – for solving the linear system (1.1). We also establish its convergence and study the choices of the optimal parameters.

Throughout this paper, we use the following notation: $\mathbb{R}^l$ will mean the usual l-dimensional Euclidean space. $\|\cdot\|$ will mean the usual 2−norm. For any $l \times l$ symmetric and positive definite matrix G, $\|x\|_G = \sqrt{\langle Gx, x\rangle} = \|G^{\frac{1}{2}} x\|$ for all $x \in \mathbb{R}^l$. For each element $v \in \mathbb{R}^{n+m}$, we will write it as $v = (v_1^T, v_2^T)^T$, where $v_1 \in \mathbb{R}^n$ and $v_2 \in \mathbb{R}^m$. And the energy-norm $|||\cdot|||$ is defined as follows (see [24–26]):

$$|||v|||^2 = \|v_1\|_{A^{-1}}^2 + \|v_2\|_K^2,$$

where $K = B^T A^{-1} B$ is the Schur complement matrix of (1.1). For any $D \in \mathbb{R}^{l\times l}$, we write $\Lambda(D)$, $\mathcal{N}(D)$ to denote the spectrum, the null space of the matrix D, respectively.

2. Convergence analysis

In this section, we will establish the convergence of the PTU iterative method (1.5) and show the choices of the optimal parameters.

By some direct calculation, (1.5) can be equivalent to

$$z_{k+1} = H(\omega,\tau) z_k + L(\omega,\tau) c,$$

where $z_k = (x_k^T, y_k^T)^T$, $c = (f^T, g^T)^T$,

$$H(\omega,\tau) = \widehat{M}(\omega,\tau)^{-1}\widehat{N}(\omega,\tau) = \begin{pmatrix} (1-\omega) I_n & -\omega A^{-1} B \\ (1-\omega)[\tau Q^{-1} B^T - (B^T B)^{-1} B^T A] & \omega(I_m - \tau Q^{-1} B^T A^{-1} B) \end{pmatrix}, \tag{2.1}$$

and

$$L(\omega,\tau) = \widehat{M}(\omega,\tau)^{-1} \begin{pmatrix} I & 0 \\ B^T & -\tau B^T B Q^{-1} \end{pmatrix} = \begin{pmatrix} \omega A^{-1} & 0 \\ (1-\omega)(B^T B)^{-1} B^T + \omega\tau Q^{-1} B^T A^{-1} & -\tau Q^{-1} \end{pmatrix}. \tag{2.2}$$

Then the PTU iterative method (1.5) is convergent if and only if the spectral radius of the iteration matrix $H(\omega, \tau)$ is less than 1. Therefore, we need to study the spectrum of $H(\omega, \tau)$ in the first place.