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Remarks on global regularity for the 3D MHD system with damping

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ABSTRACT

We investigate the Cauchy problem for the 3D MHD system with damping terms $|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}$ and $|\boldsymbol{b}|^{\beta-1}\boldsymbol{b}$ $(\alpha,\ \beta\geq 1)$, and show that the strong solution exists globally and uniquely if one of the following four conditions holds, (1) $3\leq\alpha\leq\frac{27}{8}$, $\beta\geq 4$; (2) $\frac{27}{8}<\alpha\leq\frac{7}{2}$, $\beta\geq\frac{7}{2\alpha-5}$; (3) $\frac{7}{2}<\alpha<4$, $\beta\geq\frac{5\alpha+7}{2\alpha}$; (4) $\alpha\geq 4$, $\beta\geq 1$. This improves the previous results significantly.

1. Introduction

This paper studies the following Cauchy problem for the three-dimensional (3D) incompressible magneto-hydrodynamic (MHD) equations with damping terms,

$$\begin{cases}\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}-\Delta\boldsymbol{u}+|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+\nabla\pi=\boldsymbol{0},\\ \partial_t\boldsymbol{b}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}-\Delta\boldsymbol{b}+|\boldsymbol{b}|^{\beta-1}\boldsymbol{b}=\boldsymbol{0},\\ \nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{b}=0,\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0,\quad \boldsymbol{b}|_{t=0}=\boldsymbol{b}_0,\end{cases}\tag{1}$$

where $\boldsymbol{u}$ is the fluid velocity field, $\boldsymbol{b}$ is the magnetic field, π is a scalar pressure, and $\boldsymbol{u}_0$, $\boldsymbol{b}_0$ are the prescribed initial data satisfying $\nabla\cdot\boldsymbol{u}_0=\nabla\cdot\boldsymbol{b}_0=0$ in the sense of distributions. In the damping terms, $\alpha,\ \beta\geq 1$; and if $\alpha=1$ (resp. $\beta=1$), we actually mean there is no velocity (resp. magnetic) damping.

The damping comes from the resistance to the motion of the flow. It describes various physical phenomena such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [1] and references cited therein). When $\boldsymbol{b}=\boldsymbol{0}$, system (1) reduces to the Navier–Stokes system with damping,

$$\begin{cases}\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\Delta\boldsymbol{u}+|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+\nabla\pi=\boldsymbol{0},\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0.\end{cases}\tag{2}$$

Cai and Jiu [1] first established the global existence of strong solutions when $\alpha\geq 7/2$, and the strong solution is unique in case $7/2\leq\alpha\leq 5$. This was technically improved by Zhang et al. [7], where the lower bound 7/2 was decreased to be 3. This "3" seems to be critical in some sense, and was verified by Zhou [9], where the following three results were obtained:

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(1) global strong solutions when $\alpha \geq 3$, by utilizing the following key fact:

$$\left\| \frac{|\boldsymbol{u}|^2}{|\boldsymbol{u}|^{\alpha-1}+1} \right\|_{L^\infty} \leq 1, \forall\ \alpha \geq 3; \tag{3}$$

(2) uniqueness of strong solution for any $\alpha \geq 1$, by observing the following non-negativity property:

$$\int_{\mathbb{R}^3} [|\boldsymbol{u}_1|^{\alpha-1}\boldsymbol{u}_1 - |\boldsymbol{u}_2|^{\alpha-1}\boldsymbol{u}_2] \cdot [\boldsymbol{u}_1 - \boldsymbol{u}_2] \mathrm{d}x \geq 0; \tag{4}$$

(3) fundamental regularity criteria involving $\boldsymbol{u}$ and $\nabla \boldsymbol{u}$ for $1 \leq \alpha < 3$.

For the damped MHD system (1), Ye [6] first presented the definition of weak solution, see also [3–5].

Definition 1. The triplet $(\boldsymbol{u}, \boldsymbol{b}, \pi)$ is called a weak solution of system (1) if the following three conditions are satisfied:

(1) $\boldsymbol{u} \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \cap L^{\alpha+1}(0,T;L^{\alpha+1}(\mathbb{R}^3))$,
$\boldsymbol{b} \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \cap L^{\beta+1}(0,T;L^{\beta+1}(\mathbb{R}^3))$;

(2) for any $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R}^3 \times [0,T))$ with $\nabla \cdot \boldsymbol{\phi} = 0$,

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \partial_t \boldsymbol{\phi} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \nabla \boldsymbol{u} : \nabla \boldsymbol{\phi} \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^3} [(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + |\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}] \cdot \boldsymbol{\phi} \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^3} \boldsymbol{u}_0 \cdot \boldsymbol{\phi}(x,0)\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^3} \boldsymbol{b} \cdot \partial_t \boldsymbol{\phi} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \nabla \boldsymbol{b} : \nabla \boldsymbol{\phi} \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^3} [(\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + |\boldsymbol{b}|^{\beta-1}\boldsymbol{b}] \cdot \boldsymbol{\phi} \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^3} \boldsymbol{b}_0 \cdot \boldsymbol{\phi}(x,0)\mathrm{d}x, \end{aligned}$$

where for two 3×3 matrices A, B, the notation $A : B = \sum_{i,j=1}^3 a_{ij}b_{ij}$;

(3) $\nabla \cdot \boldsymbol{u}(x,t) = \nabla \cdot \boldsymbol{b}(x,t) = 0$ for almost each $(x,t) \in \mathbb{R}^3 \times (0,T)$.

Then he showed the global regularity of (1) if one of the following five conditions verifies:

$$\alpha \geq 4, \quad \beta \geq 4; \tag{5}$$

$$3 < \alpha \leq \frac{7}{2}, \quad \frac{7}{2\alpha-5} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}; \tag{6}$$

$$\frac{7}{2} < \alpha < 4, \quad \frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}; \tag{7}$$

$$4 \leq \alpha \leq \frac{17}{3}, \quad \frac{5\alpha+7}{2\alpha} \leq \beta < 4; \tag{8}$$

$$\frac{17}{3} < \alpha \leq 7, \quad \frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{\alpha+5}{\alpha-3}. \tag{9}$$

This is quite unsatisfactory, and Zhang and Yang [8] have tried to improve it to be $\alpha > 3$, $\beta > 3$. However, there is a big mistake in using the Gagliardo–Nirenberg inequality. We have noticed this one year later after the paper published. In fact, in the first step in proving [8, Lemma 2.1], the superscript $\frac{2(\gamma-3)}{3\gamma-7}$ should be $\geq \frac{1}{2}$, which would imply $\gamma \geq 5$. And now, due to the following two important observations, we could indeed improve Ye [6] significantly.

(1) The weak solution not only belongs to the class $L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$, but also satisfies $\boldsymbol{u} \in L^{\alpha+1}(0,T;L^{\alpha+1}(\mathbb{R}^3))$ and $\boldsymbol{b} \in L^{\beta+1}(0,T;L^{\beta+1}(\mathbb{R}^3))$. As α or β grows larger, the weak solution obtains more regularity. So the upper bound restriction of β in (6)–(9) seems to be superfluous.

(2) As is well-known, in the regularity theory of MHD system, the velocity plays a dominant role (see [2,10]). This could also be the case for system (1). So intuitively for sufficiently large α, the velocity has enough regularity to ensure the smoothness of the solution, and we do not need any magnetic damping.

Precisely, our main result reads

Theorem 2. *Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$. Assume that one of the following four conditions holds:*

$$3 \leq \alpha \leq \frac{27}{8}, \quad \beta \geq 4; \tag{10}$$