Applied Mathematics and Computation 334 (2018) 174–191

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Conjugate gradient least squares algorithm for solving the generalized coupled Sylvester-conjugate matrix equations

Jingjing Hu, Changfeng Ma*

School of Mathematics and Informatics & FJKLMAA, Fujian Normal University, Fuzhou 350117, PR China

ARTICLE INFO

Keywords:
Generalized Sylvester-conjugate equations
Conjugate gradient least squares algorithm
Exact solution
Minimum-norm least squares solution

ABSTRACT

In this study, we consider the minimum-norm least squares solution of the generalized coupled Sylvester-conjugate matrix equations by conjugate gradient least squares algorithm. When the system is consistent, the exact solution can be obtained. When the system is inconsistent, the least squares solution can be obtained within finite iterative steps in the absence of round-off error for any initial matrices. Furthermore, we can get the minimum-norm least squares solution by choosing special types of initial matrices. Finally, some numerical examples are given to demonstrate the algorithm considered is quite effective in actual computation.

1. Introduction

In this paper, we consider the following generalized coupled Sylvester-conjugate matrix equations:

$$\begin{cases} A_1X + B_1Y = D_1\overline{X}E_1 + F_1, \\ A_2Y + B_2X = D_2\overline{Y}E_2 + F_2. \end{cases} \tag{1.1}$$

where $A_1, A_2, B_1, B_2, D_1, D_2 \in \mathbf{C}^{p\times m}$, $E_1, E_2 \in \mathbf{C}^{n\times n}$, $F_1, F_2 \in \mathbf{C}^{p\times n}$ are given constant matrices, while $X, Y \in \mathbf{C}^{m\times n}$ are matrices to be determined. Matrix equations are often encountered in many areas of computational mathematics [1–3,5,11,12,14,50], such as control and system theory [4,7,8], stability theory and some fields of pure and applied mathematics [6,9,10,13,15,52]. Owing to their important applications, matrix equations have attracted considerable attention from many researchers [16–24]. When the system (1.1) is consistent, by using the Kronecker product, we can transform the system (1.1) into the linear equations $Ax = b$. According to the properties of the linear equations, we can also obtain the necessary and sufficient conditions for existence and uniqueness of solution for the system (1.1). However, in order to solve the equivalent forms, the inversion of the associated large matrix need be involved, which leads to computational difficulty because excessive computer memory is required. With the increase of the sizes of the related matrices, the iterative methods have replaced the direct methods and become the main strategy for solving the matrix equations [45–49,51]. Based on the iterative solutions of matrix equations, Ding and Chen presented the hierarchical gradient iterative algorithms for general matrix equations [25,26] and hierarchical least squares iterative algorithms for generalized coupled Sylvester matrix equations and general coupled matrix equations [27,28]. An iterative algorithm for solving the coupled matrix equations $AYB = E, CYD = F$ over generalized centro-symmetric matrices was proposed [29]. By using the hierarchical identification principle and the gradient iterative method of the simple matrix equations, the gradient-based iterative algorithms were established for

* Corresponding author.
E-mail address: macf@fjnu.edu.cn (C. Ma).

https://doi.org/10.1016/j.amc.2018.03.119
0096-3003/

$AXB+CYD=F, AXB+CX^TD=F$ [30,31] and for the coupled matrix equations $A_1XB_1=F_1$ and $A_2XB_2=F_2$, this method was extend to solving some more complicated matrix equations [32–36]. In recent years, Dehghan and Hajarian considered the generalized coupled Sylvester matrix equations $AXB+CXD=M,\ EXF+GYH=N$ [37] and presented a modified conjugate gradient method to solve the generalized coupled Sylvester matrix equations over the generalized bisymmetric matrix pair (X, Y). Hajarian considered extending the CGLS algorithm for least squares solutions of the generalized Sylvester-transpose matrix equations [48]. Xie and Ma proposed a modified conjugate gradient method to solve the reflexive or anti-reflexive solutions of the following problem [38]:

$$\begin{cases} AXB+CY^TD=S_1, \\ EX^TF+GYH=S_2. \end{cases} \tag{1.2}$$

They proved the solutions can be obtained within finite iterative steps in the absence of round-off error for any initial given reflexive or anti-reflexive matrices as the system (1.1) is consistent. However, as the system (1.1) is inconsistent, how to obtain the least squares solution and the minimum-norm least squares solution is still open. Other structured matrix equations have been proposed for the problem in which the coefficient matrices are complex matrices. Dehghani-Madiseh and Dehghan [39] studied generalized solution sets of the interval generalized Sylvester matrix equation

$$\sum_{i=1}^{p} A_iX_i + \sum_{j=1}^{q} Y_jB_j = C \tag{1.3}$$

and some approaches for inner and outer estimations. Wu et al. [42] considered the solutions to the following problem

$$\sum_{\eta=1}^{p} A_{i\eta}X_\eta B_{i\eta} + C_{i\eta}\overline{X_\eta}D_{i\eta} = F_i, \quad i=1,2,\ldots,N, \tag{1.4}$$

where N is a positive integer.

It is known that the conjugate gradient method is the most popular iterative method for solving the system of linear equations

$$Ax=b, \tag{1.5}$$

where $x\in\mathbf{R}^n$ is an unknown vector, $A\in\mathbf{R}^{m\times n}$ is a given constant matrix, and $b\in\mathbf{R}^m$ is a given vector. By the definition of the Kronecker product, matrix equations can be transformed into system (1.5), then the conjugate gradient method can be applied to various linear matrix equations [37,41]. Based on this idea and improved by the aforementioned matrix equation, in this paper, we use the conjugate gradient least squares method to solve the solution of the system (1.1), as the system is inconsistent and verify that least squares solution can be obtained within finite iterative steps in the absence of round-off error for any initial matrices. Furthermore, we show that the minimum-norm least squares solution (X^*, Y^*) can be obtained by choosing special types of initial matrices. Finally, we give some numerical examples to illustrate the behavior of the algorithm considered.

Problem 1. Given $A_1, A_2, B_1, B_2, D_1, D_2\in\mathbf{C}^{p\times m}$, $E_1, E_2\in\mathbf{C}^{n\times n}$, $F_1, F_2\in\mathbf{C}^{p\times n}$, find the matrices $X\in\mathbf{C}^{m\times n}$ and $Y\in\mathbf{C}^{m\times n}$ such that

$$\left\| \begin{pmatrix} A_1X+B_1Y-D_1\overline{X}E_1-F_1 \\ A_2Y+B_2X-D_2\overline{Y}E_2-F_2 \end{pmatrix} \right\| = \min. \tag{1.6}$$

Problem 2. Let $\mathbf{S_E}$ denote the set of solution pair of Problem 1. For given matrices $(\widetilde{X},\widetilde{Y})$, find $(\widehat{X},\widehat{Y})\in\mathbf{S_E}$ such that

$$\|\widehat{X}-\widetilde{X}\|^2+\|\widehat{Y}-\widetilde{Y}\|^2 = \min\left(\|X-\widetilde{X}\|^2+\|Y-\widetilde{Y}\|^2\right). \tag{1.7}$$

The remainder of this paper is organized as follows. In Section 2, we propose a conjugate gradient least squares algorithm for the system (1.1) and give some properties. When the system is inconsistent, the least squares solution can be obtained within finite iterative steps in the absence of round-off error for any initial matrices. Furthermore, we prove that the minimum-norm least squares solution can be obtained by choosing special types of the initial matrices in Section 3. In Section 4, some numerical examples are given to demonstrate the algorithm considered is quite effective in actual computation. Finally, we give our conclusions in Section 5.

2. Conjugate gradient least squares method

2.1. Preliminaries

The following notations, definitions and lemmas will be used to develop the proposed work. We use A^T, $\mathcal{R}(A)$ and $\mathrm{tr}(A)$ to denote the transpose, the column space and the trace of a matrix A, respectively. We denote the set of all $m\times n$ real matrices, the set of all $m\times n$ complex matrices by $\mathbf{R}^{m\times n}$, $\mathbf{C}^{m\times n}$, respectively. For any matrices $A, B\in\mathbf{R}^{m\times n}$, $A\otimes B$ is their Kronecker product. For matrices A, B and X with appropriate dimension, we can easily obtain the following result:

$$\mathrm{vec}[AXB] = (B^T\otimes A)\mathrm{vec}[X].$$