Applied Mathematics and Computation 333 (2018) 8–19

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Accelerating the convergence speed of iterative methods for solving nonlinear systems[☆]

Xiao-Yong Xiao[a,b,*], Hong-Wei Yin[a,b]

[a] Department of Mathematics, School of Sciences, Nanchang University, Nanchang 330031, China
[b] Numerical Simulation and High-Performance Computing Laboratory, School of Sciences, Nanchang University, Nanchang 330031, China

ARTICLE INFO

MSC:
41A25
65H10
65Y04

Keywords:
Systems of nonlinear equations
Modified Newton method
Order of convergence
Higher order methods
Computational efficiency

ABSTRACT

In this paper, for solving systems of nonlinear equations, we develop a family of two-step third order methods and introduce a technique by which the order of convergence of many iterative methods can be improved. Given an iterative method of order $p \geq 2$ which uses the extended Newton iteration as a predictor, a new method of order $p+2$ is constructed by introducing only one additional evaluation of the function. In addition, for an iterative method of order $p \geq 3$ using the Newton iteration as a predictor, a new method of order $p+3$ can be extended. Applying this procedure, we develop some new efficient methods with higher order of convergence. For comparing these new methods with the ones from which they have been derived, we discuss the computational efficiency in detail. Several numerical examples are given to justify the theoretical results by the asymptotic behaviors of the considered methods.

1. Introduction

The construction of iterative methods for solving systems of nonlinear equations is important in numerical analysis and applied scientific branches. For a given nonlinear system $F(x)$: $D \subseteq R^n \to R^n$, we consider the problem of solving $F(x) = 0$ with n equations and n unknowns, that is, finding a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ such that $F(\alpha) = 0$, where

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^t, \quad \text{and} \quad x = (x_{(1)}, x_{(2)}, \ldots, x_{(n)})^t.$$

The most widely used algorithm for this problem is the classical Newton's method [17] when the function F is continuously differentiable and the initial approximation x_0 is close enough to α. It is given by

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \quad k = 0, 1, 2, \ldots, \tag{1.1}$$

where $F'(x)^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$.

So far, a great number of efficient methods have been proposed to improve the order of convergence and reduce the computational cost in the literature; see, for example, [3–13,16,20–22] and references therein. In numerical analysis, one should also take account of computational cost when developing new iterative methods. A general rule is achieving as high as possible convergence order requiring as small as possible the evaluations of functions, derivatives and matrix inversions.

☆ This work was supported by the National Natural Science Foundation of China (Grant no. 61563033).

* Corresponding author at: Department of Mathematics, School of Sciences, Nanchang University, Nanchang 330031, China.
E-mail address: xiaoxy31@gmail.com (X.-Y. Xiao).

https://doi.org/10.1016/j.amc.2018.03.108
0096-3003/

In recent years, some researchers also payed their attention to the discovery of general laws for accelerating the convergence of iterative methods. For solving a nonlinear equation, Kou et al.[15] made a third order method be a fifth order one by the construction

$$x_{k+1} = u_{k+1} - f(u_{k+1})/f'(y_k),$$

where $y_k = x_k - f(x_k)/f'(x_k)$, and $u_{k+1} = g_3(x_k)$ is the iteration function of a third order method. Cordero et al. [2] extended it to solve nonlinear systems. They introduced a technique by which the order of convergence $p (p \geq 2)$ of any given iterative method using the Newton iteration as a predictor can be improved to $p+2$ by the procedure

$$\begin{aligned} z_k &= \phi(x_k, y_k), \\ x_{k+1} &= z_k - F'(y_k)^{-1}F(z_k), \end{aligned} \tag{1.2}$$

where $y_k = x_k - F'(x_k)^{-1}F(x_k)$ is the classical Newton iteration, and $z_k = \phi(x_k, y_k)$ is the iteration function of a method of order p.

It is apparent that the procedure (1.2) can be applied only when the Newton iteration is used as a predictor. However, there are plenty of iterative methods which use the extended Newton iteration $y_k = x_k - aF'(x_k)^{-1}F(x_k)$ for $a \neq 0$ as a predictor. For example, Sharma and Arora [20] proposed a fourth order method defined by

$$\begin{aligned} y_k &= x_k - \frac{2}{3}F'(x_k)^{-1}F(x_k), \\ z_k &= x_k - \left[\frac{23}{8}I - F'(x_k)^{-1}F'(y_k)\left(3I - \frac{9}{8}\, F'(x_k)^{-1}F'(y_k)\right)\right]F'(x_k)^{-1}F(x_k), \end{aligned} \tag{1.3}$$

and extended it to be a sixth order method by

$$x_{k+1} = z_k - \left(\frac{5}{2}I - \frac{3}{2}F'(x_k)^{-1}F'(y_k)\right)F'(x_k)^{-1}F(z_k) \tag{1.4}$$

where y_k and z_k are defined in (1.3).

On the other hand, when the extended Newton iteration is used as a predictor, a technique has been introduced in [23]

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k), \\ z_k &= \phi(x_k, y_k), \\ x_{k+1} &= z_k - \left\{2\left[\frac{1}{2a}F'(y_k) + \left(1 - \frac{1}{2a}\right)F'(x_k)\right]^{-1} - F'(x_k)^{-1}\right\}F(z_k), \end{aligned} \tag{1.5}$$

by which the order of convergence of any given iterative method can be improved from p to $p+2$ for $p \geq 2$. Also, there are similar results given in [24,25]. Next, we want to explore other techniques to accelerate the convergence of iterative methods.

This paper is organized as follows. In Section 2, we develop a family of two-step third order methods, and introduce a technique to improve the order of convergence for any given iterative method using the extended Newton iteration as a predictor. In Section 3, some efficient methods with higher order of convergence are extended by this technique, and comparisons about computational efficiency are made. Several numerical examples are given in Section 4 to verify the theoretical results presented in Section 3. Finally, a brief conclusion is made in Section 5.

2. Main result

Recently, Sharma et al. [21] introduced a fourth order method given by

$$\begin{aligned} y_k &= x_k - \frac{2}{3}F'(x_k)^{-1}F(x_k), \\ z_k &= x_k - \frac{1}{2}\left[-I + \frac{9}{4}F'(y_k)^{-1}F'(x_k) + \frac{3}{4}F'(x_k)^{-1}F'(y_k)\right]F'(x_k)^{-1}F(x_k). \end{aligned} \tag{2.1}$$

Notice that the second step in (2.1) can be rewritten as

$$z_k = x_k - \frac{1}{2}\left[\frac{9}{4}F'(y_k)^{-1} + \frac{3}{4}F'(x_k)^{-1}F'(y_k)F'(x_k)^{-1} - F'(x_k)^{-1}\right]F(x_k).$$

Inspired by the fourth order method (2.1), we want to develop similar methods by replacing $F'(x_k)^{-1}F'(y_k)F'(x_k)^{-1}$ with $F'(y_k)^{-1}F'(x_k)F'(y_k)^{-1}$, and introduce the following construction:

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k), \\ z_k &= x_k - [bF'(y_k)^{-1} + cF'(y_k)^{-1}F'(x_k)F'(y_k)^{-1} + dF'(x_k)^{-1}]F(x_k), \end{aligned} \tag{2.2}$$

where the constants $a \neq 0$ and b, c, d are to be specified later.