Applied Mathematics and Computation 334 (2018) 192–205

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Shortcut sets for the locus of plane Euclidean networks

José Cáceres[a], Delia Garijo[b,*], Antonio González[c], Alberto Márquez[b], María Luz Puertas[a], Paula Ribeiro[d]

[a] Departamento de Matemáticas, Universidad de Almería, Ctra. Sacramento s/n, 04120 Almería, Spain
[b] Departamento de Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes s/n, 41012 Sevilla, Spain
[c] Departamento de Matemáticas, Universidad de Zaragoza, C/ Pedro Cerbuna 12, 50009 Zaragoza, Spain
[d] Instituto Superior de Engenharia, Universidade do Algarve, Estr. da Penha 139, 8005-139 Faro, Portugal

A R T I C L E I N F O

Keywords:
Euclidean network
Diameter
Shortcut set
Network design
Augmentation problem

A B S T R A C T

We study the problem of augmenting the locus $\mathcal{N}_\ell$ of a plane Euclidean network $\mathcal{N}$ by inserting iteratively a finite set of segments, called *shortcut set*, while reducing the diameter of the locus of the resulting network. There are two main differences with the classical augmentation problems: the endpoints of the segments are allowed to be points of $\mathcal{N}_\ell$ as well as points of the previously inserted segments (instead of only vertices of $\mathcal{N}$), and the notion of diameter is adapted to the fact that we deal with $\mathcal{N}_\ell$ instead of $\mathcal{N}$. This increases enormously the hardness of the problem but also its possible practical applications to network design. Among other results, we characterize the existence of shortcut sets, compute them in polynomial time, and analyze the role of the convex hull of $\mathcal{N}_\ell$ when inserting a shortcut set. Our main results prove that, while the problem of minimizing the size of a shortcut set is NP-hard, one can always determine in polynomial time whether inserting only one segment suffices to reduce the diameter.

1. Introduction

A *geometric network* of points in the plane is an undirected graph whose vertices are points in $\mathbb{R}^2$ and whose edges are straight-line segments connecting pairs of points. When edges are assigned lengths equal to the Euclidean distance between their endpoints, the geometric network is called *Euclidean network*. These networks are the objects of study in this paper; concretely, we deal with *plane* Euclidean networks in which edges may intersect only at their endpoints. For simplicity and when no confusion may arise, we shall simply say network, it being understood as plane Euclidean network and, unless otherwise stated, networks are assumed to be connected.

It is natural to distinguish between a network $\mathcal{N}$ and its locus, denoted by $\mathcal{N}_\ell$, where one is considering not only the vertices of $\mathcal{N}$ but the set of all points of the Euclidean plane that are on $\mathcal{N}$ (thus, $\mathcal{N}_\ell$ is treated indistinctly as a network or as a closed point set). Roughly speaking, in order to compute the *distance* between two vertices of $\mathcal{N}$ one has to sum the lengths of the edges of a shortest path connecting them, and the *diameter* of $\mathcal{N}$ is the maximum among the distances between any two vertices. This concept is naturally extended to $\mathcal{N}_\ell$; the difference is that now the distances that must be computed are not only between vertices but also between two arbitrary points along the edges of $\mathcal{N}$. In this setting, a

* Corresponding author.
E-mail addresses: jcaceres@ual.es (J. Cáceres), dgarijo@us.es (D. Garijo), gonzalezh@unizar.es (A. González), almar@us.es (A. Márquez), mpuertas@ual.es (M.L. Puertas), pribeiro@ualg.pt (P. Ribeiro).

https://doi.org/10.1016/j.amc.2018.04.010
0096-3003/

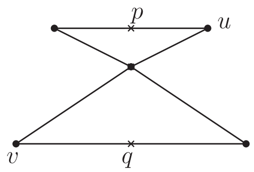

Fig. 1. The distance between u and v gives the diameter of the network, but the diameter of its locus is given by the distance between p and q.

shortest path connecting two points of $\mathcal{N}_\ell$ may consist of a number of edges and one or two fragments of edges (in which the points to be connected are located). When computing the distance, we sum the lengths of all the edges in the path and the lengths of the fragments[1]. See Fig. 1.

In this paper, we study the following problem:

Problem 1. Given a plane Euclidean network $\mathcal{N}$, insert a finite set of segments $\mathcal{S} = \{s_1, \ldots, s_k\}$ in order to reduce (or minimize) the diameter of the locus of the resulting network, provided that the endpoints of segment s_1 are on $\mathcal{N}_\ell$ and the endpoints of s_i, $2 \leq i \leq k$, are on $\mathcal{N}_\ell \cup \{s_1, \ldots, s_{i-1}\}$. (We say that $\mathcal{S}$ is a *shortcut set* for $\mathcal{N}_\ell$.)

When a shortcut set $\mathcal{S}$ is inserted into the network, there may be crossings between the segments in $\mathcal{S}$, and also between those segments and the network edges. In this paper, we consider that every crossing creates a new network vertex, which is a model that is applicable to a wide range of situations, for example, in urban network design: to introduce shortcut sets is a way of improving the worst-case travel time along networks of roads in a city, highways, etc. Note that such models are considering the locus of the network, which is also used for related applications to location analysis [4] and feed-link problems [4–6]. The modification or transformation of a network can be applied or models many other different situations. For instance, very recently, Li et al. [7] have proposed a rule for deleting not only edges but nodes of a complex network in order to illustrate its collapsing behavior, and in [8], the authors use the local structure of a network to identify influential spreaders with all the social applications that this identification allows. One can find many applications of Euclidean networks to robotics, telecommunications networks, computer networks, and flight scheduling. Any collection of objects that have some connections between them can be modeled as a geometric network. See [9–11] for more details and references.

Problem 1 also belongs to the class of *graph augmentation* problems. Concretely, it is a variant of the Diameter-Optimal k-Augmentation Problem for edge-weighted geometric graphs, where one has to insert k additional edges (i.e., a shortcut set of size k with segments connecting vertices) to an edge-weighted plane geometric graph in order to minimize the diameter of the resulting graph. Whilst there are numerous studies on the non-geometric version of this problem (see for instance [12–14]), when the input is an edge-weighted plane geometric graph much less is known, not only on this problem but on graph augmentation in general (see the survey [15] for results and references on graph augmentation over plane geometric graphs). Farshi et al. [16] designed approximation algorithms for the problem of adding one edge to a plane Euclidean network in $\mathbb{R}^d$ while minimizing the dilation. The same problem is considered in [17,18] but for networks embedded in a metric space.

When considering the locus $\mathcal{N}_\ell$ instead of the network $\mathcal{N}$ (setting of Problem 1), the hardness of the problem is enormously increased which motivates that there are very few results on this topic, all restricted to specific families of graphs. Yang [19] studied the restriction of Problem 1 to inserting only one segment, called *shortcut*, to the locus of a simple polygonal chain.[2] Among other results, he designed three different approximation algorithms to compute an *optimal* shortcut, i.e., a shortcut that minimizes the diameter among all shortcuts. De Carufel et al. [2,20] and Bae et al. [21] consider a variant of Problem 1 in which, in contrast with our model, the crossings that may appear when inserting shortcuts do not form new network vertices, and the endpoints of all their shortcuts are on $\mathcal{N}_\ell$. A linear time algorithm to find an optimal shortcut for paths is provided in [2], and also optimal pairs of shortcuts (i.e., $k = 2$) for convex cycles. Trees have been studied in [20] where the authors give an algorithm to find an optimal shortcut for a tree of size n in $O(n \log n)$ time. Finally, for circles, Bae et al. [21] have analyzed how to add up to seven shortcuts in an optimal way.

Our main contribution in this paper is to provide the first approach to Problem 1 for general plane Euclidean networks. Concretely, in Section 2, we first characterize the networks $\mathcal{N}$ whose locus $\mathcal{N}_\ell$ admits a shortcut set. For such networks, we find in polynomial time a shortcut set for $\mathcal{N}_\ell$ giving an upper bound on its size. The section concludes by studying the connection between the diameter of the convex hull of $\mathcal{N}_\ell$ and the diameter of the resulting network when inserting a shortcut set to $\mathcal{N}_\ell$.

Section 3 is devoted to prove that it is always possible to determine in polynomial time whether inserting only one segment to $\mathcal{N}_\ell$ suffices to reduce the diameter. Our method computes such a segment in case of existence, and combines a

[1] The diameter of $\mathcal{N}_\ell$ is the *generalized diameter* of $\mathcal{N}$, which was introduced in [1] and called *continuous diameter* in, for example [2,3], but we use the context of *locus* because it fits better to our purpose in this paper.

[2] In fact, the restriction of Problem 1 to inserting only one segment to the locus of a general plane Euclidean network was proposed by Cai as a personal communication to Yang [19].