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Maximal classes of matrices determining generalized inverses

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ABSTRACT

This paper derives some further results on recent generalized inverses studied in the literature, namely core EP, DMP, and CMP inverses. Our main aim is to develop maximal classes of matrices for which their representations remain valid.

1. Introduction

Let $\mathbb{C}^{n\times n}$ be the set of $n \times n$ complex matrices. For $A \in \mathbb{C}^{n\times n}$, the symbols A^*, A^{-1}, $\text{rk}(A)$, $\mathcal{N}(A)$, and $\mathcal{R}(A)$ will denote the conjugate transpose, the inverse, the rank, the kernel, and the range space of A, respectively. Moreover, I_n will refer to the $n \times n$ identity matrix.

Let $A \in \mathbb{C}^{n\times n}$. We recall that the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$(1)\, AXA = A, \quad (2)\, XAX = X, \quad (3)\, (AX)^* = AX, \quad \text{and} \quad (4)\, (XA)^* = XA,$$

is called the Moore–Penrose inverse of A and is denoted by $A^\dagger$.

For a given complex square matrix A, the index of A, denoted by $\text{Ind}(A)$, is the smallest nonnegative integer k such that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. We observe that the index of a nonsingular matrix A is 0, and by convention, the index of the null matrix is 1.

We also recall that the Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ such that

$$(2)\, XAX = X, \quad (5)\, AX = XA, \quad \text{and} \quad (6^k)\, A^{k+1}X = A^k,$$

where $k = \text{Ind}(A)$, and is denoted by A^d. If $A \in \mathbb{C}^{n\times n}$ satisfies $\text{Ind}(A) \le 1$, then the Drazin inverse of A is called the group inverse of A and is denoted by $A^\#$.

A matrix $X \in \mathbb{C}^{n\times n}$ that satisfies the only equality $AXA = A$ is called an inner inverse or $\{1\}$-inverse of A, and a matrix $X \in \mathbb{C}^{n\times n}$ that satisfies the unique equality $XAX = X$ is called an outer inverse or $\{2\}$-inverse of A. In general, the set of matrices satisfying the conditions (i), (j), ... is denoted by $A\{i, j, \dots\}$. A matrix $X \in A\{i, j, \dots\}$ is called an $\{i, j, \dots\}$-inverse of A, and is denoted by $A^{(i,j,\dots)}$. Let $P_A := AA^\dagger$ and $Q_A := A^\dagger A$ denote the orthogonal projectors onto the range of A and the range of A^*, respectively. Similarly, $P_{A^\ell} := A^\ell (A^\ell)^\dagger$ and $Q_{A^\ell} := (A^\ell)^\dagger A^\ell$ for any positive integer ℓ.

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The core inverse was introduced by Baksalary and Trenkler in [2] and later investigated by S. Malik in [7] and S.Z. Xu, J.L. Chen, X.X. Zhang in [15], among others. For a given matrix $A \in \mathbb{C}^{n\times n}$, it is defined as the unique matrix $X \in \mathbb{C}^{n\times n}$ that satisfies the conditions

$$AX = P_A \qquad \text{and} \qquad \mathcal{R}(X) \subseteq \mathcal{R}(A).$$

In case that such a matrix X exists, it is denoted by $A^{\oplus}$. Moreover, it was proved that A is core invertible if and only if $\mathrm{Ind}(A) \le 1$.

Two generalizations of the core inverse were introduced for $n \times n$ complex matrices, namely core EP inverses and DMP inverses. In order to recall these concepts we assume that $\mathrm{Ind}(A) = k$ for a given matrix $A \in \mathbb{C}^{n\times n}$. Firstly, the unique matrix $X \in \mathbb{C}^{n\times n}$ such that

$$XAX = X \qquad \text{and} \qquad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}\left(A^k\right), \tag{1.1}$$

is called the core EP inverse of A and is denoted by $A^{\textcircled{\dagger}}$ [11]. In [11, Lemma 3.3] K.M. Prasad and K.S. Mohana proved that the core EP of a matrix $A \in \mathbb{C}^{n\times n}$ is the unique solution of

$$XA^{k+1} = A^k, \quad XAX = X, \quad (AX)^* = AX, \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k). \tag{1.2}$$

Recently, in [4, Theorem 2.7], it is was proved that the core EP inverse can be expressed as

$$A^{\textcircled{\dagger}} = (AP_{A^k})^{\dagger}. \tag{1.3}$$

Secondly, the concept of DMP inverse of A was introduced in [8] by S. Malik and N. Thome. In this case, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X, \qquad XA = A^dA, \quad \text{and} \quad A^kX = A^kA^{\dagger}, \tag{1.4}$$

is called the DMP inverse of A and is denoted by $A^{d,\dagger}$. Moreover, it was proved that

$$A^{d,\dagger} = A^dAA^{\dagger}. \tag{1.5}$$

In addition, a new generalized inverse was investigated in [9] by M. Mehdipour and A. Salemi. In this case, the matrix

$$A^{c,\dagger} = Q_AA^dP_A, \tag{1.6}$$

is called the CMP inverse of A.

The well-known Urquhart formula [1, p. 48] establishes that

$$A^{\dagger} = A^{(1,4)}AA^{(1,3)}, \tag{1.7}$$

that is it allows us to represent the Moore–Penrose inverse by means of any {1, 4}-inverse and any {1, 3}-inverse of A. Similarly, it occurs with the Mitra [10] and Zlobec [16] formula given by

$$A^{\dagger} = A^*YA^*,$$

for arbitrary $Y \in (A^*AA^*)\{1\}$ and the Decell formula [3, Corollary 1] given by

$$A^{\dagger} = A^*XAYA^*, \tag{1.8}$$

where X and Y are any element in $(AA^*)\{1\}$ and $(A^*A)\{1\}$, respectively. Related to the group inverse, it is well known [1, p. 168] that

$$A^{\#} = AXA,$$

where X is an arbitrary element in $(A^3)\{1\}$, or more generally, for the Drazin inverse it holds

$$A^d = A^{\ell}XA^{\ell},$$

for ℓ being any integer not less than the index of A, and X being an arbitrary element in $(A^{2\ell+1})\{1\}$.

The common fact is that all these formulas are represented by matrices in a more general class. Our main aim is to develop maximal classes for the most recent generalized inverses investigated in the literature, namely core EP, DMP, and CMP inverses.

For any matrix $A \in \mathbb{C}^{n\times n}$ of rank $r > 0$, the Hartwig–Spindelböck decomposition is given by

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*, \tag{1.9}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is a diagonal matrix, the diagonal entries σ_i being singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$.

On the other hand, the following two results were recently obtained by H. Kurata in [6].

Proposition 1.1 *[6, Theorem 3]. Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index 1 written as in (1.9). Let $r = \mathrm{rk}(A)$ and $X, Y \in A\{1\}$. The following conditions are equivalent:*