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Partial-approximate controllability of nonlocal fractional evolution equations via approximating method

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ABSTRACT

In this paper we study partial-approximate controllability of semilinear nonlocal fractional evolution equations in Hilbert spaces. By using fractional calculus, variational approach and approximating technique, we give the approximate problem of the control system and get the compactness of approximate solution set. Then new sufficient conditions for the partial-approximate controllability of the control system are obtained when the compactness conditions or Lipschitz conditions for the nonlocal function are not required. Finally, we apply our abstract results to the partial-approximate controllability of the semilinear heat equation and delay equation.

1. Introduction

Controllability concepts are important properties of a control system that plays a substantial role in many engineering problems, such as stabilizing unstable systems using feedback control. Therefore, in recent years, controllability problems for various types of linear and semilinear (fractional) dynamical systems have been studied in many articles (see [1–31] and the references therein). From the mathematical point of view it is necessary to distinguish between problems of exact and approximate controllability. Exact controllability allows to move the system to an arbitrary final state, while approximate controllability means that the system can be moved to an arbitrary small neighborhood of the final state. In particular, approximately controlled systems are more predominant, and very often the approximate controllability is quite adequate in applications. There are some interesting and important approximate controllability results for semilinear evolution systems in abstract spaces with a Caputo fractional derivative. Sakthivel et. al. [23] initiated to study approximate controllability of fractional differential systems in Hilbert spaces. Meanwhile, Sakthivel and Ren [24], Debbouche and Torres [8], Mahmudov [15,16] pay attention to the study of approximate controllability for various types of (fractional) evolution systems in abstract spaces. On the other hand, existence and approximate controllability of nonlocal fractional equations were studied in [11,12,14,17]

This paper is devoted to the study of partial-approximate controllability of the semilinear fractional evolution equation with nonlocal conditions,

$$\begin{aligned} &{}^{C}D_t^q y(t) = Ay(t) + Bu(t) + f(t, y(t)),\ \ 0 < t \le T, \\ &y(0) = y_0 - g(y), \end{aligned} \tag{1}$$

where the state variable $y(\cdot)$ takes values in the Hilbert space X, ${}^{C}D_t^q$ is the Caputo fractional derivative of order q with $\frac{1}{2} < q \le 1$, A: $D(A) \subset X \to X$ is a family of closed and bounded linear operators generating a strongly continuous semigroup

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$S:[0,b]\to\mathfrak{L}(X)$, where the domain $D(A)\subset X$ is dense in X, the control function $u(\cdot)$ is given in $L^2([0, b], U)$, U is a Hilbert space, B is a bounded linear operator from U into X, f: $[0, b]\times X\to X$, g: $C([0, b], X)\to X$ are given functions satisfying some assumptions to be specified later and y_0 is an element of the Hilbert space X.

One of the purposes of this article is to investigate the partial-approximate controllability of the system (1) without Lipschitz continuous or compact assumptions on the non-local term g. In fact, g is supposed to be continuous and is completely defined by $[\delta, b]$ for some small $\delta>0$. Meanwhile, in order to obtain the existence of solutions of the control system (1), we construct an approximate problem of the system (1) and obtain the compactness of the set of approximate solutions. It differs from the usual approach that the fixed-point theorem is applied directly to the corresponding solution operator. Our results, therefore, can be seen as the extension and development of existing results.

It has not yet been reported work on the partial-approximate controllability of fractional semilinear evolution equations with nonlocal conditions. Inspired by the aforementioned recent contributions, we propose to discuss the partial-approximate controllability of fractional evolution systems in Hilbert spaces with classical nonlocal conditions. We first impose a partial-approximate controllability of the associated linear system. Then we develop a variational approach in [16,17] and approximating method in [14], and rewrite our control problem as a sequence of fixed point problems. Next using the Schauder fixed point theorem we get the existence of fixed points and show that these solutions (fixed points) steers the system to an arbitrary small neighborhood of the final state in a closed subspace.

We organize the article as follows. In Section 2, we provide preliminaries, assumptions and formulate the main result on partial-approximate controllability. In Section 3, we extend the variational method to construct approximating control for the approximating controllability problem and use the Schauder fixed point theorem to show existence of a solution for the approximating controllability problem. The partial-approximate controllability result is proved in Section 4. In this section we solve the difficulty concerning the compactness of solution operator $S(t)$ at $t=0$ by means of approximate solution set constructed in Section 3. The difficulty is due to the fact that a compact semigroup $S(t)$ is not compact at $t=0$ in an infinite dimensional space. Finally, we provide two examples to illustrate the application of the abstract results.

2. Statement of results

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ be the set of positive integers, real numbers and positive real numbers, respectively. We denote by X a Hilbert space with norm $\|\cdot\|$, $C([0, b], X)$ the space of all X-valued continuous functions on $[0, b]$ with the norm $\|\cdot\|_C$, $L^2([0, b], U)$ the space of all U-valued square integrable functions on $[0, b]$, $\mathcal{L}(X)$ the space of all bounded linear operators from X to X with the usual norm $\|\cdot\|_{\mathcal{L}(X)}$, let A be the infinitesimal generator of C_0-semigroup $\{S(t)\colon t\geq 0\}$ of uniformly bounded linear operators on X. Clearly, $M_S:=\sup\left\{\|S(t)\|_{\mathcal{L}(X)}: t\geq 0\right\}<\infty$. Let E be a closed subspace of X and denote by Π the projection from X onto E.

We recall some notations, definitions and results on fractional derivative and fractional differential equations.

Definition 1 [22]**.** The Riemann–Liouville fractional order derivative of f: $[0,\infty)\to X$ of order $q\in\mathbb{R}_+$ is defined by

$$^{RL}D^q f(t)=\frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-q-1}f(s)ds,$$

where $q\in(n-1,n)$, $n\in\mathbb{N}$.

Definition 2 [22]**.** The Caputo fractional order derivative of f: $[0,\infty)\to X$ of order $q\in\mathbb{R}_+$ is defined by

$$^{RL}D^q f(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}f^{(n)}(s)ds,$$

where $q\in(n-1,n)$, $n\in\mathbb{N}$.

Now, we use the probability density function to give the following definition of mild solutions to (1).

Definition 3 [28]**.** A solution $y(\cdot;u)\in C([0, b], X)$ is said to be a mild solution of (1) if for any $u\in L^2([0, b], U)$ the integral equation

$$y(t)=S_q(t)(y_0-g(y))+\int_0^t (t-s)^{q-1}T_q(t-s)[Bu(s)+f(s,y(s))]ds,\quad 0\leq t\leq b, \tag{2}$$

is satisfied. Here

$$S_q(t)=\int_0^\infty \omega_\alpha(\theta)S(t^\alpha\theta)d\theta,\quad T_q(t)=\alpha\int_0^\infty \theta\omega_\alpha(\theta)S(t^\alpha\theta)d\theta,\quad t\geq 0,$$

$$\omega_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-1/\alpha}\varpi_\alpha\left(\theta^{-1/\alpha}\right)\geq 0,$$

$$\varpi_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\quad \theta\in(0,\infty),$$