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Asymptotic representation of solutions for second-order impulsive differential equations

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ABSTRACT

We obtain sufficient conditions which guarantee the existence of a solution of a class of second order nonlinear impulsive differential equations with fixed moments of impulses possessing a prescribed asymptotic behavior at infinity in terms of principal and nonprincipal solutions. An example is given to illustrate the relevance of the results.

1. Introduction

The asymptotic integration problem for second-order ordinary differential equations is a classical research topic in mathematics. It has been widely investigated by many authors for the last several decades, see for instance [1–10] and the references cited therein. The problem is to find sufficient conditions to guarantee the existence of a solution with a prescribed behavior at infinity. It turns out that the asymptotic integration of impulsive differential equations is in its early stages. Among many difficulties are: (i) one cannot employ the techniques of asymptotic integration theory available for ordinary differential equations due to difficulties caused by impulsive perturbations, and (ii) the solutions of linear homogeneous equations even in the simplest case cannot be calculated in a closed form to figure out principal and nonprincipal solutions. To the best of our knowledge the study in [11] is the only one in which a constructive approach to the asymptotic integration problem for second-order impulsive differential equations is given. In the present work we aim at contributing the problem further by studying the impulsive differential equations of the form

$$\begin{cases} (p(t)x')' + q(t)x = f(t,x), & t \neq \theta_i, \quad i = 1,2,\ldots, \\ \Delta p(t)x' + q_i x = \tilde{f}_i(x), & t = \theta_i. \end{cases} \tag{1}$$

It is shown that under some mild conditions there are solutions of Eq. (1) with prescribed asymptotic behaviors at infinity. The behavior is shown to be closely related to certain solutions called principal and nonprincipal solutions of certain linear homogeneous impulsive differential equations. More specifically, we will show that for any given $a_1, a_2, b_2, a_3 \in \mathbb{R}$, Eq. (1) has solutions $x_1(t)$, $x_2(t)$, $x_3(t)$ possessing the following asymptotic properties:

$$x_1(t) = a_1 v(t) + o(v(t)), \quad t \to \infty,$$

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$$x_2(t) = a_2 v(t) + b_2 u(t) + o(u(t)), \quad t \to \infty,$$
$$x_3(t) = a_3 v(t) + o((u(t))^{\lambda}(v(t))^{1-\lambda}), \quad t \to \infty, \quad \text{for some } \lambda \in (0,1),$$

where u and v are principal and nonprincipal solutions of

$$\begin{cases} (p(t)x')' + q(t)x = 0, & t \neq \theta_i, \quad i = 1, 2, \ldots, \\ \Delta p(t)x' + q_i x = 0, & t = \theta_i. \end{cases} \tag{2}$$

It should be noted that such results prove very useful for studying certain boundary value problems on the half line [14].

With regard to Eq. (1) we assume that $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, $\tilde{f}_i \in C(\mathbb{R}, \mathbb{R})$ and $\{\theta_i\}$ is a strictly increasing sequence of real numbers such that $\lim_{i\to\infty} \theta_i = \infty$, $\{q_i\}$ is a sequence of real numbers. As usual, Δ is the impulse operator defined by $\Delta y|_{t=\theta_i} = y(\theta_i+) - y(\theta_i-)$, where $y(\theta_i\pm) = \lim_{t\to\theta_i\pm} y(t)$.

By a solution of Eq. (1) we mean a continuous function x defined on $[t_0, \infty)$ so that x', $(p(t)x')'$ are piece-wise left continuous, and x satisfies Eq. (1) for $t \in [t_0, \infty)$.

2. Preliminaries

We first provide a lemma on principal and nonprincipal solutions of Eq. (2) extracted from [12]. The result is exactly the same as in ordinary differential equations case but the proof is somewhat different due to impulse effects.

Lemma 2.1. *If Eq. (2) has a positive solution, then there exist two linearly independent solutions u and v satisfying the following properties:*

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0; \tag{3}$$

$$\int_T^{\infty} \frac{1}{p(t)u^2(t)} \mathrm{d}t = \infty, \quad \int_T^{\infty} \frac{1}{p(t)v^2(t)} \mathrm{d}t < \infty; \tag{4}$$

$$\frac{p(t)v'(t)}{v(t)} > \frac{p(t)u'(t)}{u(t)}, \quad t \geq T, \tag{5}$$

where T is sufficiently large.

Here u and v are called as principal and nonprincipal solutions of Eq. (2), respectively. We note that the principal solution $u(t)$ is unique up to a constant multiple, and any solution which is linearly independent of $u(t)$ is nonprincipal. For an extensive knowledge about principal and nonprincipal solutions of ordinary differential equations we refer to [13]. It is easy to verify that if $v(t)$ is a nonprincipal solution of Eq. (2) then

$$u(t) = v(t) \int_t^{\infty} \frac{1}{p(s)v^2(s)} \mathrm{d}s \tag{6}$$

becomes a principal solution. Conversely, if $u(t)$ is the principal solution of Eq. (2) then

$$v(t) = u(t) \int_T^{t} \frac{1}{p(s)u^2(s)} \mathrm{d}s \tag{7}$$

turns out to be a nonprincipal solution.

Next we state two compactness criteria that will be utilized in our proofs.

Lemma 2.2 [15]. *Let $\Omega \in \mathbb{R}^n$. A set $M \subset L^p(\Omega), 1 \leq p < \infty$ is compact if (i) there exists a number $B > 0$ such that $\|f\|_{L^p(\Omega)} \leq B$ for all $f \in M$, and (ii) $\|(\tau_h f) - f\|_{L^p(\Omega)} \to 0$ as $h \to 0$, where $(\tau_h f)(x) := f(x_1 + h, x_2 + h, \ldots, x_n + h)$, $x \in \Omega$.*

Lemma 2.3 [16]. *Let $\Omega \in \mathbb{R}^n$. A set $M \subset l^p(\Omega)$, $1 \leq p < \infty$ is totally bounded if, and only if (i) M is pointwise bounded, and (ii) for every $\epsilon > 0$ there is some $n \in \mathbb{N}$ so that, for every $x \in M$, $\Sigma_{k>n} |x_k|^p < \epsilon^p$.*

3. Main result

In what follows by $PLC([t_0, \infty), \mathbb{R})$ we denote the set of real-valued left continuous functions defined on $[t_0, \infty)$, and $u(t)$ and $v(t)$ are the principal and nonprincipal solution of Eq. (2), respectively. Further, we assume without any loss of generality that $u(t)$ and $v(t)$ are positive on an interval $[T, \infty)$ for some $T \geq t_0$ sufficiently large.

Theorem 3.1. *Suppose that there exist functions $g_j \in C(\mathbb{R}_+, \mathbb{R}_+)$, $j = 1, 2$, $h_k \in C([t_0, \infty), \mathbb{R}_+)$, $k = 1, 2$ and a sequence of positive real numbers $\{\tilde{h}_i\}$ such that*

$$|f(t, x)| \leq h_1(t) g_1\left(\tfrac{|x|}{v(t)}\right) + h_2(t), \quad t \geq T, \tag{8}$$

$$|\tilde{f}_i(x)| \leq \tilde{h}_i g_2\left(\tfrac{|x|}{v(\theta_i)}\right), \quad \theta_i \geq T. \tag{9}$$