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Dynamical analysis of a two prey-one predator system with quadratic self interaction

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ABSTRACT

In this paper we investigate the dynamical properties of a two prey-one predator system with quadratic self interaction represented by a three-dimensional system of differential equations by using tools of computer algebra. We first investigate the stability of the singular points. We show that the trajectories of the solutions approach to stable singular points under given conditions by numerical simulation. Then, we determine the conditions for the existence of the invariant algebraic surfaces of the system and we give the invariant algebraic surfaces to study the flow on the algebraic invariants which is a useful approach to check if Hopf bifurcation exists.

1. Introduction

The stability of the ecological systems and the persistence of populations of the species within them are most important concerns of mathematical physics as well as ecology. One of the most important ecological models is the predator-prey model. The dynamic relationship between the predator and its prey has been for a long time one of the main themes in ecology due to its universal existence and importance [1]. Dynamical analysis of spatial predator-prey models where the species interact not only with the ones living in the same spatial location but also interact with the ones living in spatially adjacent locations have gained much attention [2–4]. Dynamical complexity of predator-prey models with time delay have also been extensively studied [5–8]. Reflecting these concerns, mathematical models of ecological systems have been used to investigate the stability of a variety of generalizations of the Lotka–Volterra model (see for instance, [9–12]) which is known as one of the first dynamical systems to represent the predator-prey interaction as proposed by Lotka [13] and Volterra [14]. They are among the first to study this phenomena by making a number of assumptions that led to this nontrivial but tractable mathematical problem where x and y denote the population densities of the prey and the predator evolving with time t. The Lotka–Volterra model is given as

$$\frac{dx}{dt} = x(a - by),$$
$$\frac{dy}{dt} = y(cx - d),$$

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where a is the natural growth rate of the prey population in the absence of predation, b is the death rate per encounter of the prey due to predation, c is the efficiency of turning consumed prey into predator and d is the natural death rate of the predator in the absence of food. The Lotka–Volterra model has been generalized to more advanced systems in order to represent more realistic models which possess stable singular points. The model has been mentioned with the fact that it does not contain quadratic self interaction terms. The quadratic self interaction terms serve the purpose of modeling how prey-A interacts with other prey-A and prey-B interacts with other prey-B. The underlying idea for deploying quadratic self interaction terms only in prey equations is that if $x = u^2$, $y = v^2$ and $z = w^2$, then there are interaction couplings such as uv^2 and u^2v and vw^2 and v^2w in the prey and predator equations [15]. Self interaction terms denote competition within species and therefore is shown by negative terms in the prey equations. In this paper we enhance this idea of employing quadratic self interaction terms in the prey equations to three-dimensional systems.

Two prey-one predator system with quadratic self interaction in the prey equations where one predator species interacts with two teams of prey species is considered in this work. If the population densities of prey-A and prey-B are denoted by x and z, respectively and the population density of the predator is denoted by y, then the two prey-one predator system with quadratic self interaction is given by

$$\begin{aligned}\frac{dx}{dt} &= x(a_1 - b_1x - c_1y),\\ \frac{dy}{dt} &= y(-a_2 + b_2x + c_2z),\\ \frac{dz}{dt} &= z(a_3 - b_3y - c_3z),\end{aligned} \tag{1}$$

where a_1 and a_3 are growth rates of prey-A and prey-B in the absence of the predator, a_2 is the death rate of the predator in the absence of the preys, c_1 and b_3 are the death rates of prey-A and prey-B due to predation, b_2 and c_2 are the consumption rates of the predator over prey-A and prey-B and b_1 and c_3 are quadratic self interaction rates of prey-A and prey-B. All parameters and variables are assumed to be nonnegative values in order to have physical explanations. System (1) was proposed in [16] where the goal of work was monitoring the corresponding species in a class of predator-prey systems in order to analyze the population dynamics. In this work a nonlinear auxiliary system (so called the observer system) is designed to reconstruct the unknown states or unmeasurables theoretically by employing high order polynomials. Although the method of the observer is powerful, there is an estimation error. Therefore, it is necessary to study the system by using qualitative methods. In this paper, we present our results on the system by using the methods of stability and algebraic invariants to understand the behavior of the system in detail.

One of the major tasks of the dynamical systems theory is to analyze the qualitative properties of a dynamical system which is defined by ODEs. Indeed, one might try to solve this problem by using numerical simulations but using them gives information only on a particular solution and not on the qualitative behavior of the dynamical system. However, the most useful aspect of dynamical systems theory is that one can predict some features of the phase portrait of a system governed by ODEs without actually solving the system. The simplest examples of such information are the number and the positions of singular points. Also the stability of a singular point can be detected without solving the system by local stability analysis.

A more advanced method to understand the qualitative behavior of a dynamical system consists of finding the invariant algebraic surfaces. This powerful algebraic approach helps us to find points where Hopf bifurcation can occur when local methods fail to give results. It is also very useful to find algebraic first integrals [17,18]. For these reasons, the method has been applied to many important problems of mathematical physics such as the Lorenz system [19–23] and various generalizations of the Lotka–Volterra model [24,25].

In this paper we first perform a local stability analysis of system (1), i.e. we find the singular points and study their stability in Section 2. In Section 3 we describe an approach to find the invariant algebraic surfaces of the polynomial systems of ODEs and use it to find all such surfaces of degree one and two of system (1) in order to understand qualitative properties of the system. Then, in Section 4 we perform a study of the flow on the invariant plane of a subfamily of system (1).

2. The stability of the singular points

In this section we give results on the positions and the stability of the singular points of nonlinear system (1). Before giving results on stability of the singular points, we give the following information on the invariant set of system (1).

System (1) has two first integrals

$$I_1 = 1 + x + \frac{c_1}{b_1 + b_2}y - \frac{c_3}{b_1}z$$

and

$$I_2 = x^{-b_1b_3c_2+b_2b_3c_3}y^{b_1(b_3+c_1)c_3}z^{c_1(b_1c_2-b_2c_3)}.$$

System (1) has the invariant set

$$\left\{(x, y, z)|1 + x + \frac{c_1}{b_1 + b_2}y - \frac{c_3}{b_1}z = h\right\}$$