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Integral representations of two generalized core inverses

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ABSTRACT

In this paper, we present integral representations for the DMP and core-EP inverse, which based on the full-rank decomposition of a given matrix. In particular, integral representations of the core and dual core inverse are given. All of these integral representations do not require any restriction on the spectrum of a certain matrix.

1. Introduction

In 1977, Groetsch [9] introduced an integral representation of the Moore–Penrose inverse $T^{\dagger}$ of a bounded linear operator $T \in \mathcal{L}(H_1, H_2)$ with closed range $\mathcal{R}(T)$ in Hilbert spaces,

$$T^{\dagger} = \int_0^{\infty} \exp(-T^*Tt)T^* dt,$$

where H_1, H_2 are Hilbert spaces.

In 2002, Castro González et al. [4] provided an integral representation of the Drazin inverse a^D in Banach algebras: let $a \in \mathcal{A}$ be a Drazin invertible element of a finite Drazin index $k \geq 1$ such that the nonzero spectrum of a^{m+1} lies in the open right half of the complex plane for some $m \geq k$,

$$a^D = \int_0^{\infty} \exp(-a^{m+1}t)a^m dt.$$

However, this integral representation was restricted to the spectrum of the element a^{m+1}. Later, they eliminated this drawback by establishing the integral representation for the Drazin inverse A^D in [5]. Namely, suppose that $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. Then

$$A^D = \int_0^{\infty} \exp[-tA^k(A^{2k+1})^*A^{k+1}]A^k(A^{2k+1})^*A^k dt.$$

Let T be a subspace of $\mathbb{C}^n$ and let S be a subspace of $\mathbb{C}^m$. The generalized inverse $A^{(2)}_{T,S}$ [2] of matrix $A \in \mathbb{C}^{m\times n}$ is the matrix $G \in \mathbb{C}^{n\times m}$ satisfying

$$GAG = G, \mathcal{R}(G) = T, \mathcal{N}(G) = S.$$

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It is known that the six kinds of classical generalized inverses: the Moore–Penrose inverse $A^{\dagger}$, the weighted Moore–Penrose inverse $A^{\dagger}_{M,N}$, the group inverse $A^{\#}$, the Drazin inverse A^{D}, the Bott–Duffin inverse $A^{(-1)}_{(L)}$ and the generalized Bott–Duffin inverse $A^{(\dagger)}_{(L)}$ can be presented as particular generalized inverses $A^{(2)}_{T,S}$ with prescribed range and null space (see [2,3,7,14]).

In 2003, Wei [17] established an unified integral representation for the generalized inverse $A^{(2)}_{T,S}$ according to the above mentioned results. Let $A \in \mathbb{C}^{m\times n}$ and $G \in \mathbb{C}^{n\times m}$ with $\mathcal{R}(G) = T$ $\mathcal{N}(G) = S$, the range and null spaces respectively of G. If nonzero eigenvalues λ of GA satisfy $Re\lambda > 0$, then

$$A^{(2)}_{T,S} = \int_0^{\infty} \exp(-GAt)Gdt.$$

Meanwhile, Wei and Djordjević [18] also established a general integral representation for the generalized inverse $A^{(2)}_{T,S}$ which drops the restriction on the spectrum of GA. The general integral representation is

$$A^{(2)}_{T,S} = \int_0^{\infty} \exp(-G(GAG)^*GAt)G(GAG)^*Gdt,$$

which extended the result of Drazin inverse [5].

In 2010, the core and dual core inverse were introduced by Baksalary and Trenkler for square matrices of index at most 1 in [1]. In 2014, the core inverse was extended from two different perspectives. One is the DMP inverse, defined by Malik and Thome, for square matrices of an arbitrary index in [11]. The other is the core-EP inverse introduced by Manjunatha Prasad and Mohana [12]. In this paper, the dual conceptions of the DMP and core-EP inverse were called the MPD and dual core-EP inverse, respectively. Various characterizations and representations of the DMP, core-EP and core inverse have been investigated [6,8,10,13,15,19,20] in complex matrices, operators and rings.

From the above mentioned integral representations for the classical generalized inverses, we wonder to know if the DMP, core-EP and core inverse have respective integral representation.

The aim of this paper is to obtain integral representations of the DMP and core-EP inverse, which based on the full-rank decomposition of a given matrix. The corresponding integral representations of their dual inverses are also derived. In particular, integral representations of the core and dual core inverse are given. All of these integral representations do not require any restriction on the spectrum of a certain matrix.

2. Preliminaries

Definition 2.1 [2]**.** Let $A \in \mathbb{C}^{m\times n}$. The unique matrix $A^{\dagger} \in \mathbb{C}^{n\times m}$ is called the Moore–Penrose inverse of A if it satisfies

$$AA^{\dagger}A = A, \ A^{\dagger}AA^{\dagger} = A^{\dagger}, \ (AA^{\dagger})^* = AA^{\dagger}, \ (A^{\dagger}A)^* = A^{\dagger}A.$$

For arbitrary matrix $A \in \mathbb{C}^{n\times n}$, $A^*, \mathcal{R}(A)$ and rk(A) denote its conjugate transpose, column space and rank, respectively. We know that if k is the least positive integer such that rk(A^{k+1})=rk(A^k), then it is called the index of A and is denoted by $ind(A)$.

Definition 2.2 [2]**.** Let $A \in \mathbb{C}^{n\times n}$. The unique matrix $A^{D} \in \mathbb{C}^{n\times n}$ is called the Drazin inverse if it satisfies

$$A^{k+1}A^D = A^k, \ A^DAA^D = A^D, \ AA^D = A^DA,$$

where $k = ind(A)$. When $k = 1$, the Drazin inverse reduced to the group inverse and it is denoted by $A^{\#}$.

Definition 2.3 [11]**.** Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index k (not necessarily ≤ 1). The unique matrix $A^{D,\dagger}$ is called the DMP inverse of A if it satisfies

$$A^{D,\dagger}AA^{D,\dagger} = A^{D,\dagger}, \ A^{D,\dagger}A = A^DA, \ A^kA^{D,\dagger} = A^kA^{\dagger}.$$

Dually, if there exists a matrix $X \in \mathbb{C}^{n\times n}$ satisfies the following equations:

$$XAX = X, \ AX = AA^D, \ XA^k = A^{\dagger}A^k.$$

Then X is called the MPD inverse of A and is denoted by $A^{\dagger,D}$.

Definition 2.4 [1]**.** A matrix $A^{\#\!\!\!\!\bigcirc} \in \mathbb{C}^{n\times n}$ matrix is called the core inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies

$$AA^{\#\!\!\!\!\bigcirc} = P_A \text{ and } \mathcal{R}(A^{\#\!\!\!\!\bigcirc}) \subseteq \mathcal{R}(A).$$

Dually, A matrix $A_{\#\!\!\!\!\bigcirc} \in \mathbb{C}^{n\times n}$ is called the dual core inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies

$$A_{\#\!\!\!\!\bigcirc}A = P_{A^*} \text{ and } \mathcal{R}(A_{\#\!\!\!\!\bigcirc}) \subseteq \mathcal{R}(A^*).$$

Definition 2.5 [12]**.** A matrix X, denoted by $A^{\dagger\!\!\!\!\bigcirc}$, is called the core-EP inverse of A if it satisfies

$$XAX = X \ and \ \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^D).$$