Applied Mathematics and Computation 334 (2018) 356–366

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A characterization theorem for semi-classical orthogonal polynomials on non-uniform lattices

A. Branquinho[a], Y. Chen[b], G. Filipuk[c], M.N. Rebocho[a,d,*]

[a] CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, Coimbra 3001-501, Portugal
[b] Faculty of Science and Technology, Department of Mathematics, University of Macau, Av. Padre Tomás Pereira, Taipa, Macau, China
[c] Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, Warsaw 02–097, Poland
[d] Departamento de Matemática, Universidade da Beira Interior, Covilhã 6201-001, Portugal

A R T I C L E I N F O

MSC:
33C45
33C47
42C05

Keywords:
Orthogonal polynomials
Divided-difference operator
Non-uniform lattices
Askey–Wilson operator
Semi-classical class

A B S T R A C T

It is proved a characterization theorem for semi-classical orthogonal polynomials on non-uniform lattices that states the equivalence between the Pearson equation for the weight and some systems involving the orthogonal polynomials as well as the functions of the second kind. As a consequence, it is deduced the analogue of the so-called compatibility conditions in the ladder operator scheme. The classical orthogonal polynomials on non-uniform lattices are then recovered under such compatibility conditions, through a closed formula for the recurrence relation coefficients.

1. Introduction

Semi-classical orthogonal polynomials on special non-uniform lattices (snul) are related to a divided difference operator, say $\mathbb{D}$, whose support is the so-called q-quadratic lattice [15,19]. Under some specifications, $\mathbb{D}$ is the Askey-Wilson operator [1]. Such families of orthogonal polynomials are well-know within the theory of discrete orthogonal polynomials, and find many applications within a vast list of topics from Mathematical Physics (see, amongst many others, [9,15–17,19]).

In the classification of lattices and corresponding divided difference operators (see [19, Section 2, Table 2] and [12]), the q-quadratic lattices are a generalization of other lower complexity lattices, such as the quadratic, q-linear and linear lattices. Such a hierarchy of lattices is related to the well-known q-Askey scheme [10].

The main motivation for this paper comes from some properties that characterize semi-classical orthogonal polynomials in the continuous setting, the so-called structure relations, that is, difference-differential relations connecting two consecutive polynomials,

$$AP_n' = L_nP_n + M_nP_{n-1}, \tag{1}$$

or, in view of the three-term recurrence relation,

$$AP_n' = \tilde{L}_nP_n + \tilde{M}_nP_{n+1}, \tag{2}$$

with $A, L_n, M_n, \tilde{L}_n, \tilde{M}_n$ polynomials of degree independent of n (the degree of P_n). The classification of orthogonal polynomials via such kind of equations has a long history, see, for instance, [14]. On a more general framework, (1)–(2) are the lowering

* Corresponding author at: Departamento de Matemática, Universidade da Beira Interior, Covilhã 6201-001, Portugal.
E-mail addresses: ajplb@mat.uc.pt (A. Branquinho), yayangchen@umac.mo (Y. Chen), filipuk@mimuw.edu.pl (G. Filipuk), mneves@ubi.pt (M.N. Rebocho).

https://doi.org/10.1016/j.amc.2018.04.022
0096-3003/

and raising relations, deduced in the ladder operator scheme [6]. Similar equations to (1)–(2), with the derivative replaced by difference operators, are well-known in the literature (see, for instance, the introduction of [11] and references therein). For the snul case, see [12,19].

In the present paper we give a characterization of semi-classical orthogonal polynomials on snul via difference systems that involve the polynomials as well some related functions, the so-called functions of the second kind (see Section 2 for more details). Combining those systems with the three-term recurrence relation we then deduce difference equations in the matrix form, giving some fundamental relations that we regard as the discrete analogue of the ones appearing in the ladder operator scheme [6]. Here, we would like to put emphasis on the formula for the determinant in Corollary 2. Through such a formula, we obtain a closed form equation for the recurrence relation coefficients of the classical families of orthogonal polynomials on snul [8].

The remainder of the paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we show the main results of the paper, Theorem 1 and Corollaries 1 and 2. The compatibility conditions are discussed in Section 3.1. In Section 4 we show the formulae for the recurrence relation coefficients of the classical orthogonal polynomials on snul.

2. Preliminary results

We consider the divided difference operator $\mathbb{D}$ given as in [12, Eq. (1.1)], with the property that $\mathbb{D}$ leaves a polynomial of degree $n-1$ when applied to a polynomial of degree n. The operator $\mathbb{D}$, defined on the space of arbitrary functions, is given in terms of two functions y_1, y_2 (at this stage, unknown),

$$(\mathbb{D}f)(x) = \frac{f(y_2(x)) - f(y_1(x))}{y_2(x) - y_1(x)}. \tag{3}$$

The functions y_1, y_2 may be obtained as follows: applying $\mathbb{D}$ to $f(x)=x^2$ and $f(x)=x^3$, one obtains, respectively,

$$y_1(x) + y_2(x) = \text{polynomial of degree } 1, \tag{4}$$

$$(y_1(x))^2 + y_1(x)y_2(x) + (y_2(x))^2 = \text{polynomial of degree } 2, \tag{5}$$

the later condition being equivalent to $y_1(x)y_2(x) =$ polynomial of degree less or equal than 2. Hence, conditions (4)–(5) define y_1 and y_2 as the two y-roots of a quadratic equation

$$\hat{a}y^2 + 2\hat{b}xy + \hat{c}x^2 + 2\hat{d}y + 2\hat{e}x + \hat{f} = 0, \quad \hat{a} \neq 0. \tag{6}$$

Set $\lambda = \hat{b}^2 - \hat{a}\hat{c}$, $\tau = ((\hat{b}^2 - \hat{a}\hat{c})(\hat{d}^2 - \hat{a}\hat{f}) - (\hat{b}\hat{d} - \hat{a}\hat{e})^2)/\hat{a}$.

If $\lambda \neq 0$, as y_1, y_2 are the roots of (6), we have

$$y_1(x) = p(x) - \sqrt{r(x)}, \quad y_2(x) = p(x) + \sqrt{r(x)}, \tag{7}$$

with p, r polynomials given by

$$p(x) = -\frac{\hat{b}x + \hat{d}}{\hat{a}}, \quad r(x) = \frac{\lambda}{\hat{a}^2}\left(x + \frac{\hat{b}\hat{d} - \hat{a}\hat{e}}{\lambda}\right)^2 + \frac{\tau}{\hat{a}\lambda}. \tag{8}$$

The q-quadratic lattices correspond to the case $\lambda\tau \neq 0$ [2,12,15,16]. There is a well-known parametrization of the conic (6), say $x = x(s), y = y(s)$, such that

$$y_1(x) = x(s-1/2), \quad y_2(x) = x(s+1/2),$$

given as [2,12,15]

$$x(s) = \kappa_1 q^s + \kappa_2 q^{-s} + \kappa_3,$$

for some appropriate constants κ's, and q defined through

$$q + q^{-1} = \frac{4\hat{b}^2}{\hat{a}\hat{c}} - 2, \quad q \neq 1. \tag{9}$$

Note that, in this case, we have the divided-difference operator (3) given as

$$\mathbb{D}f(x(s)) = \frac{f(x(s+1/2)) - f(x(s-1/2))}{x(s+1/2) - x(s-1/2)}. \tag{10}$$

In such a case, the polynomials p, r are then recovered under

$$x(s+1/2) + x(s-1/2) = 2p(x(s)), \quad (x(s+1/2) - x(s-1/2))^2 = 4r(x(s)). \tag{11}$$