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Fully discrete spectral methods for solving time fractional nonlinear Sine–Gordon equation with smooth and non-smooth solutions

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A B S T R A C T

We consider the initial boundary value problem of the time fractional nonlinear Sine–Gordon equation and the fractional derivative is described in Caputo sense with the order $\alpha(1<\alpha<2)$. Two fully discrete schemes are developed based on Legendre spectral approximation in space and finite difference discretization in time for smooth solutions and non-smooth solutions, respectively. Numerical stability and convergence are analysed. Numerical experiments for both the fully discrete schemes are presented to confirm our theoretical analysis.

1. Introduction

Fractional differential equations (FDEs) have been found more realistic in modelling a variety of physical phenomena, engineering process, biological system and financial products, such as anomalous diffusion and non-exponential relaxation patterns [1]. Typically, such scenarios involve long-range temporal cumulative memory effects and/or long-range spatial interactions that can be more accurately described by fractional-order models [2–5].

Much attention has been gained in FDEs, finding numerical methods to solve FDEs especially for the fractional calculus has been popular in the twenty-first century, such as finite difference/finite element methods [6–15], spectral method or spectral collocation method [16–21], variational iteration method [22,23], spline collocation method [24], and other numerical methods [25–27].

In this paper, we consider time fractional Sine–Gordon equation

$$ {}_0^C D_t^{\alpha} u - \partial_x^2 u + \sin u = f(x,t), \qquad x \in (-1,1), \quad 0 < t \le T, \tag{1.1} $$

subject to the initial condition

$$ u(x,0) = g_0(x), \quad u_t(x,0) = g_1(x) \quad x \in [-1,1], \tag{1.2} $$

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and boundary condition

$$u(-1,t)=0,\quad u(1,t)=0,\quad 0<t\le T, \tag{1.3}$$

where α is a parameter describing the fractional order of the time derivative in the Caputo sense and $1<\alpha<2$, when $\alpha=2$, (1.1) reduces to the classical Sine–Gordon equation. g_0, g_1 are given functions. The Caputo definition with fractional-order derivative $1<\alpha<2$ is defined as

$$ {}_0^C D_t^{\alpha} u(x,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\partial^2 u(x,s)}{\partial s^2}\frac{ds}{(t-s)^{\alpha-1}}.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Sine–Gordon equation is a very important nonlinear hyperbolic equation and used widely in many fields of physics including applications in the chain of coupled pendulums and modelling the propagation of transverse electromagnetic wave on a superconductor transmission system. For fractional Sine–Gordon equation the authors of [28] showed that the dynamics on the 1D lattice can be equivalently presented by the corresponding fractional nonlinear equation in the long-wave limit and they concentrated on the conditions of such equivalence, type of the equations with fractional derivatives and some related properties. As examples of their developments, fractional Sine–Gordon and wave-Hilbert nonlinear equations have been found for classical lattice dynamics.

Sine–Gordon equation has been studied by many researchers using different numerical methods, for example, integer Sine–Gordon equation has been solved by variational iteration method [29], spectral method [30], space fractional Sine–Gordon equation has been solved by compact difference scheme [31], time fractional Sine–Gordon equation by implicit RBF meshless approach [32]. We use Legendre spectral method for solving time fractional Sine–Gordon equation due to the high accuracy of spectral method, and we get the temporal convergence order is $3-\alpha$, this result is better than [32] which the convergence rate in time is first order. Since the analytical solutions to the FDEs are not smooth in most cases, such as the solutions in time to the FDEs are like t^{σ}, where σ is not an integer. Then we give a fully discrete scheme for solving non-smooth solutions, we follow Lubich's correction approach by adding the correction terms, for more details see [33,34], the numerical results show better performance compared to the fully discrete scheme we give.

An outline of this paper is as follows. We commence by reviewing some preliminaries and notations and recall the result of Legendre projection approximation in Section 2. In Section 3, we give the time discretization and build a linear fully discrete Legendre spectral scheme. We prove the unconditionally stability and convergence in Section 4, respectively. In Section 5, we present numerical experiments to confirm the effectiveness of our scheme and numerical results also confirm the fully discrete scheme we give for non-smooth solutions. In Section 6, some numerical examples are given. Finally, we make a conclusion.

2. Preliminaries

Denote $\Lambda=(-1,1)$. Let $L^2(\Lambda)$, $L^\infty(\Lambda)$, and $H^m(\Lambda)$ be the usual Sobolev spaces equipped with norms $\|\cdot\|$, $\|\cdot\|_\infty$ and $\|\cdot\|_m$. The inner product of $L^2(\Lambda)$ and $H^m(\Lambda)$ are denoted by $(\cdot,\cdot)$ and $(\cdot,\cdot)_m$. $|\cdot|_m$ denotes the semi-norm of $H^m(\Lambda)$. Furthermore,

$$H_0^1(\Lambda)=\{v\in H^1(\Lambda)|v(\pm 1)=0\}.$$

We denote by $L^\infty(0,T;H^m(\Lambda))$ the space of the measurable functions u: $[0,T]\to H^m(\Lambda)$, such that

$$\|u\|_{L^\infty(H^m)}=\operatorname*{ess\,sup}_{0\le t\le T}\|u(t)\|_m<\infty.$$

Denote $C^k([0,T];H^m(\Lambda))(0\le k<\infty)$ the space of $k-$ times continuous differentiable functions $v:[0,T]\to H^m(\Lambda)$, such that

$$\|u\|_{C^k(H^m)}=\sum_{i=0}^{k}\max_{0\le t\le T}\|v^{(i)}(t)\|_m<\infty.$$

Throughout this paper c is a generic positive constant independent of N.

Let N be a positive integer. We denote by $P_N(\Lambda)$ be the space of all polynomials of degree less than or equal to N. $P_N^0:=\{u\in P_N|u(\pm 1)=0\}$. Now we recall some projection approximation results.

Let $\pi_N^{1,0}$ be the H_0^1- orthogonal projection operator from $H_0^1(\Lambda)$ into P_N^0, such that for all $u\in H_0^1(\Lambda)$,

$$(\partial_x\pi_N^{1,0}u,\partial_x v)=(\partial_x u,\partial_x v),\quad v\in P_N^0.$$

For the projection operator $\pi_N^{1,0}$, one has the following approximation result:

Lemma 2.1 [35]. *For all $u\in H_0^1(\Lambda)\cap H^{\min\{N,m\}}(\Lambda)$, we have*

$$\|u-\pi_N^{1,0}u\|_k\le cN^{k-m}\|u\|_m,\quad k=0,1,m\ge 1,$$