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The smallest eigenvalue of large Hankel matrices

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ABSTRACT

We investigate the large N behavior of the smallest eigenvalue, λ_N, of an $(N+1)\times(N+1)$ Hankel (or moments) matrix $\mathcal{H}_N$, generated by the weight $w(x)=x^{\alpha}(1-x)^{\beta},\ x\in[0,1],\ \alpha>-1,\ \beta>-1$. By applying the arguments of Szegö, Widom and Wilf, we establish the asymptotic formula for the orthonormal polynomials $P_n(z), z\in\mathbb{C}\setminus[0,1]$, associated with $w(x)$, which are required in the determination of λ_N. Based on this formula, we produce the expressions for λ_N, for large N.

Using the parallel algorithm presented by Emmart, Chen and Weems, we show that the theoretical results are in close proximity to the numerical results for sufficiently large N.

1. Introduction

Let $\mu(x)$ be a positive measure with the bounded support $I(\subseteq\mathbb{R})$ and define the moment sequence of $\mu(x)$ by

$$h_k := \int_I x^k d\mu(x) \qquad k=0,1,2,\ldots. \tag{1.1}$$

We know that the Hankel determinant plays a significant role in the theory of random Hermitian matrices. Associated with $\mu(x)$, the $(N+1)\times(N+1)$ Hankel matrix $\mathcal{H}_N$, is defined by

$$\mathcal{H}_N := (h_{m+n})_{m,n=0}^{N}. \tag{1.2}$$

It is known that the smallest eigenvalue of the Hankel matrix is intimately related to the distribution function $\mu(x)$. We are motivated by the fact that the smallest eigenvalue depends $\mu'(x)$ in a non-trivial way.

Let $I=[a,b]$, where a and b are fixed constants, such that the Szegö condition,

$$\int_a^b \frac{\ln w(x)}{\sqrt{(b-x)(x-a)}}dx > -\infty, \tag{1.3}$$

with $w(x)=\mu'(x)$ is satisfied. The asymptotic behavior of the Hankel determinants for large enough N is given by Szegö [2–5].

Let λ_N denote the smallest eigenvalue of $\mathcal{H}_N$. The behavior of λ_N, N large, has attracted a lot of attention. See e.g. Szegö [3], Widom and Wilf [7–9], Chen et al. [10,11,15], Berg et al. [15,16], etc. Szegö [3] studied the special cases for $w(x)$, defined

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on I, which can either be a finite or infinite. For finite cases, if $w(x)=1, x\in(-1,1)$ or $w(x)=1, x\in(0,1)$, the smallest eigenvalues for large N are given, respectively, by*

$$\lambda_N \simeq 2^{\frac{9}{4}}\pi^{\frac{3}{2}}\sqrt{N}\left(\sqrt{2}+1\right)^{-2N-3},$$

$$\lambda_N \simeq 2^{\frac{15}{4}}\pi^{\frac{3}{2}}\sqrt{N}\left(\sqrt{2}+1\right)^{-4N-4}.$$

Widom and Wilf [7] found a kind of 'universal' law, where they show that if $w(x)>0, x\in[a,b]$, and the Szegö condition (1.3) is satisfied, then

$$\lambda_N \simeq A\sqrt{N}B^{-N},$$

where A and B are computable constants depending on $w(x)$, a, b, and are independent of N.

For cases of an infinite interval, Szegö [3] chose the Gaussion weight ($w(x)=\mathrm{e}^{-x^2},\ x\in\mathbb{R}$) and Laguerre weight ($w(x)=\mathrm{e}^{-x},\ x\geq 0$). The corresponding smallest eigenvalues are approximated, respectively, by

$$\lambda_N \simeq \mathrm{e}2^{\frac{13}{4}}\pi^{\frac{3}{2}}N^{\frac{1}{4}}\mathrm{e}^{-2\sqrt{2N}},$$
$$\lambda_N \simeq \mathrm{e}2^{\frac{7}{2}}\pi^{\frac{3}{2}}N^{\frac{1}{4}}\mathrm{e}^{-4\sqrt{N}}.$$

Chen and Lawrence generalized the results of Szegö in [3]. By means of Dyson's Coulomb fluid method, they deduced the case for $w(x)=\mathrm{e}^{-x^{\beta}},\ x\in[0,+\infty),\ \beta>\frac{1}{2}$ and then gave two asymptotic formulas of λ_N for $\beta=n+1/2$ and $\beta\neq n+1/2$, $n=1,2,3,\dots$, respectively. See [10] for details.

Recently, Zhu, Chen, Emmart and Weems [13] studied the asymptotic behavior of λ_N where they chose the weight $w(x)=x^{\alpha}\mathrm{e}^{-x^{\beta}}, x\in[0,\infty),\ \alpha>-1,\ \beta>\frac{1}{2}$. This generalized the work in [3,10]. In specially, they obtained the approximation formula of λ_N for the Laguerre weight $w(x)=x^{\alpha}\mathrm{e}^{-x},\ x\in[0,\infty),\ \alpha>-1$.

We note that the smallest eigenvalues of the examples given above are exponentially small. Hence, it's hard to determine the smallest eigenvalues of the Hankel matrices associated with these weights by numerical techniques.

This paper is organized as follows, firstly, we establish the asymptotic formula for the orthonormal polynomials $P_n(z)$ associated with the weight $w(x)$ in Theorem 2.1. Then in Theorem 2.2, we give the specific asymptotic expression of λ_N. Finally, we present some numerical results compared with our theoretical results in Section 3.

In order to meet the demands of some proofs in our results, we define the whole complex plane by $\mathbb{C}\cup\{\infty\}$, and the unit disc by

$$D := \{z\in\mathbb{C}\big|\,|z|\leq 1\},$$

with its boundary (unit circle)

$$\partial D := \{z\in\mathbb{C}\big|\,|z|=1\}.$$

2. Main results

In this section, we shall produce the asymptotic expression for λ_N, the smallest eigenvalue of the $(N+1)\times(N+1)$ Hankel matrix $\mathcal{H}_N$. We consider the weight

$$w(x)=x^{\alpha}(1-x)^{\beta}, x\in[0,1],\ \alpha>-1,\ \beta>-1, \tag{2.1}$$

which satisfies

$$\int_0^1 \frac{\ln w(x)}{\sqrt{x(1-x)}}dx = -2\pi(\alpha+\beta)\ln 2 > -\infty.$$

The $N+1$ by $N+1$ Hankel matrix $\mathcal{H}_N$ is defined by

$$\mathcal{H}_N := (h_{m+n})_{m,n=0}^{N},$$

where h_{m+n} is the $(m+n)$th moment with respected to $w(x)$, reads

$$h_{m+n} := \int_0^1 x^{m+n}w(x)dx = \int_0^1 x^{m+n+\alpha}(1-x)^{\beta}dx,\ \ m,n=0,1,2,\dots.$$

By the definition of the Gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1}\mathrm{e}^{-t}dt,\quad \Re\, x>0,$$

* Throughout this paper, the relation $a_n\simeq b_n$ means $\lim_{n\to\infty}a_n/b_n=1$.