Applied Mathematics and Computation 333 (2018) 225–230

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some properties of comaximal right ideal graph of a ring[☆]

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ARTICLE INFO

MSC:
05C25
16N40
16D25

Keywords:
Comaximal right ideal graph
Central idempotent
Local ring

ABSTRACT

For a ring R (not necessarily commutative) with identity, the comaximal right ideal graph of R, denoted by $\mathcal{G}(R)$, is a graph whose vertices are the nonzero proper right ideals of R, and two distinct vertices I and J are adjacent if and only if $I+J=R$. In this paper we consider a subgraph $\mathcal{G}^*(R)$ of $\mathcal{G}(R)$ induced by $V(\mathcal{G}(R))\backslash\mathcal{J}(R)$, where $\mathcal{J}(R)$ is the set of all proper right ideals contained in the Jacobson radical of R. We prove that if R contains a nontrivial central idempotent, then $\mathcal{G}^*(R)$ is a star graph if and only if R is isomorphic to the direct product of two local rings, and one of these two rings has unique maximal right ideal $\{0\}$. In addition, we also show that R has at least two maximal right ideals if and only if $\mathcal{G}^*(R)$ is connected and its diameter is at most 3, then completely characterize the diameter of this graph.

1. Introduction

Throughout this paper, R is assumed to be a ring (not necessarily commutative) with identity. Let $U(R)$ be the set of all units of R, $\mathbb{I}(R)$ be the set of all proper right ideals of R, $Max_r(R)$ be the set of all maximal right ideals of R. If $|Max_r(R)|=1$, then R is called a *local ring*. The *Jacobson radical* $J(R)$ of a ring R is defined to be the intersection of all the maximal right ideals of R. If $J(R)=\{0\}$, then R is said to be *semisimple*. A ring R is called *nil-semisimple* if it has no nilpotent ideals different from $\{0\}$. We denote $\mathcal{J}(R)=\{I|\ I$ is a right ideal of R, and $I\subseteq J(R)\}$.

We review some notions related to this paper. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set, respectively. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of edges of G incident with v. G is said to be *connected* if for any two distinct vertices u and v in G there exists a path from u to v, and G is *totally disconnected* if $E(G)=\emptyset$. The *distance* $d(u,v)$ between two vertices u and v in G is the length of a shortest (u,v)-path in G (if there is no path connecting u and v, we define $d(u,v)=\infty$). The *diameter* of G, denoted by $diam(G)$, is equal to

$$\sup\{d(u,v)|u,v\in V(G)\}.$$

The *girth* of G, denoted by $g(G)$, is the length of the shortest cycle in G. If G has no cycles, then $g(G)=\infty$. G is called a *tree* if G is connected and has no cycles. A *k-partite* graph is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset. A *complete k-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite* (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. Moreover, if $m=1$ or $n=1$, then this graph is said to be a *star graph*. A *complete* graph is one in which each pair of distinct vertices

☆ This research was supported by NSFC (No. 11671402), Hunan Provincial Natural Science Foundation (2016JJ2138, 2018JJ2479), the Fundamental Research Funds for the Central Universities of CSU (2017zzts057) and Mathematics and Interdisciplinary Sciences Project of CSU.

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https://doi.org/10.1016/j.amc.2018.03.075
0096-3003/

is joined by an edge, we use K_n to denote the complete graph with n vertices. A subset S of $V(G)$ is called a *clique* if the subgraph induced by S is complete.

Using the properties of graphs to study algebraic structures becomes an exciting topic in the last decades. In particular, using graph theory to study a ring attracts much attention. Let R be a commutative ring with identity. Sharma and Bhatwadekar [18] defined the comaximal graph $\Gamma(R)$, with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra+Rb=R$. Then they obtained a computational formula for the chromatic number of this graph. Maimani et al. [16] investigated the connectedness and the diameter of the graph $\Gamma_2(R)-J(R)$, where $\Gamma_2(R)$ is the subgraph of $\Gamma(R)$ induced by non-unit elements. Wang [21] characterized those rings R for which $\Gamma_2(R)-J(R)$ is a forest or Eulerian, and he also characterized all the finite rings R such that the genus of $\Gamma_2(R)$ (resp. $\Gamma(R)$) is at most one. Samei [17] studied the girth and dominating number of $\Gamma_2(R)-J(R)$, and obtained the algebraic and topological characterizations for graphical properties of this graph. In addition, several researchers [1,20] generalized the results from [16,21] to the non-commutative rings.

Recently, Ye and Wu [23] defined another natural graph over a commutative ring R with identity, namely, the co-maximal ideal graph, denoted by $\mathcal{C}(R)$, is a graph whose vertices are the proper ideals of R which are not contained in the Jacobson radical $J(R)$ of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1+I_2=R$. They proved that $\mathcal{C}(R)$ is a connected graph with diameter at most 3, and they also studied the clique number and chromatic number of $\mathcal{C}(R)$. Akbari et al. [2] showed that there exists a vertex of $\mathcal{C}(R)$ that adjacent to all other vertices if and only if R is isomorphic to the direct product of a local ring and a field. Ye et al. [22] used the graph blow-up method to present a complete classification of rings R whose graphs $\mathcal{C}(R)$ are non-empty planar graphs. For other related works on co-maximal ideal graphs of commutative rings, one may refer to [4,7].

In addition, Amini et al. [3] extended the above concept to a non-commutative ring R with identity. They defined a new graph on R with vertices as the nonzero proper right ideals of R, and two distinct vertices I and J are adjacent if and only if $I+J=R$. This graph is called the *comaximal right ideal graph* of non-commutative ring R, denoted by $\mathcal{G}(R)$. They characterized the rings R for which $\mathcal{G}(R)$ respectively is connected, complete, planar, complemented or a forest.

In the present paper, we consider the subgraph $\mathcal{G}^*(R)$ of $\mathcal{G}(R)$ induced by $V(\mathcal{G}(R))\backslash\mathcal{J}(R)$. In Section 2, we show that $\mathcal{G}^*(R)$ is a complete bipartite graph if and only if the cardinal number of the set $Max_r(R)$ is equal 2. Also we prove that if R contains a nontrivial central idempotent, then $\mathcal{G}^*(R)$ is a star graph if and only if $R\cong R_1\times R_2$, where each R_i is local and one of R_i has unique maximal right ideal $\{0\}$. In Section 3, we show that $|Max_r(R)|\geq 2$ if and only if $\mathcal{G}^*(R)$ is connected and $diam(\mathcal{G}^*(R))\leq 3$. Moreover, we also completely characterize the diameter of this graph.

2. Properties of $\mathcal{G}^*(R)$

In this section, we present some properties of the comaximal right ideal graph $\mathcal{G}^*(R)$. These results show that the graph $\mathcal{G}^*(R)$ has many properties similar to those of the comaximal graph $\Gamma_2(R)-J(R)$ in [16].

Proposition 2.1. *Let I be any nonzero proper right ideal of a ring R. Then $I\in\mathcal{J}(R)$ if and only if $deg_{\mathcal{G}(R)}(I)=0$.*

Proof. Let $deg_{\mathcal{G}(R)}(I)=0$ and assume that $I\notin\mathcal{J}(R)$. Then there exists $M\in Max_r(R)$ such that $I\nsubseteq M$. Therefore $I+M=R$. This contradicts our assumption.

Conversely, suppose that $deg_{\mathcal{G}(R)}(I)\neq 0$. Then there exists $J\in V(\mathcal{G}(R))$ such that $I+J=R$. On the other hand, for the proper right ideal J of R, there exists $N\in Max_r(R)$ such that $J\subseteq N$. From $I\subseteq J(R)\subseteq N$, we conclude that $I+J\subseteq N\neq R$, which is a contradiction. □

According to Proposition 2.1, we know that each $I\in\mathcal{J}(R)$ is an isolated vertex of the graph $\mathcal{G}(R)$. Thus the main part of $\mathcal{G}(R)$ is the subgraph $\mathcal{G}^*(R)$. Based on this reason the main aim of this paper is to study the properties of $\mathcal{G}^*(R)$.

Theorem 2.2. *Let R be a semisimple right Artinian ring which is not simple. Then the following are equivalent:*

(1) *$\mathcal{G}^*(R)$ is a finite graph.*
(2) *R has only finitely many right ideals.*
(3) *Every vertex of $\mathcal{G}^*(R)$ has finite degree.*

Moreover, $\mathcal{G}^*(R)$ has n vertices if and only if R has only n nonzero proper right ideals.

Proof.

(1) $\Rightarrow$(2) Let $\mathcal{G}^*(R)$ is a finite graph with n vertices. Since R is semisimple, every nonzero proper right ideal of R is a vertex of $\mathcal{G}^*(R)$. This leads to $|\mathbb{I}(R)\backslash\{0\}|=n$. Thus R has only finitely many right ideals.
(2) $\Rightarrow$(3) It is clear.
(3) $\Rightarrow$(1) By the Wedderburn–Artin Theorem [24, p.562], we have $R\cong M_{n_1}(\Delta_1)\times M_{n_2}(\Delta_2)\times\cdots\times M_{n_r}(\Delta_r)$, where each $M_{n_i}(\Delta_i)$ is a matrix ring over division ring Δ_i. Since R is not simple, $r>1$. Now we show that $M_{n_i}(\Delta_i)$ contains finitely many right ideals for $i=1,2,\ldots,r$. If $\{I_j\}_{j=1}^{\infty}$ is an infinite family of right ideals of $M_{n_1}(\Delta_1)$, for every $j\geq 1$, then the vertex $J=M_{n_1}(\Delta_1)\times\{0\}\times\cdots\times\{0\}$ of $\mathcal{G}^*(R)$ is adjacent to $I_j\times M_{n_2}(\Delta_2)\times\cdots\times M_{n_r}(\Delta_r)$. So the degree of J is not finite, this contradicts our assumption. By a similar argument, $M_{n_i}(\Delta_i)$ has finitely many right ideals for $i=2,3,\ldots,r$. This implies that R has finitely many right ideals. Furthermore, $\mathcal{G}^*(R)$ is a finite graph. □