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Iterative methods for finding commuting solutions of the Yang–Baxter-like matrix equation

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ABSTRACT

The main goal of this paper is the numerical computation of solutions of the so-called Yang–Baxter-like matrix equation $AXA = XAX$, where A is a given complex square matrix. Two novel matrix iterations are proposed, both having second-order convergence. A sign modification in one of the iterations gives rise to a third matrix iteration. Strategies for finding starting approximations are discussed as well as a technique for estimating the relative error. One of the methods involves a very small cost per iteration and is shown to be stable. Numerical experiments are carried out to illustrate the effectiveness of the new methods.

1. Introduction

In this work, we are interested in the numerical evaluation of commuting solutions X of the quadratic matrix equation

$$AXA = XAX, \tag{1}$$

where A is a given complex matrix of order n, i. e., $A \in \mathbb{C}^{n\times n}$. This equation is called the Yang–Baxter-like matrix equation (and abbreviated to YB-like equation) because of its connections with the classical Yang–Baxter equation arising in statistical mechanics. More details about those connections can be found, for instance, in [4,6,7] and the references therein.

Note that the YB-like matrix equation has at least two trivial solutions: $X = 0$ and $X = A$. Of course, the interest in solving (1) is to find nontrivial solutions. This has been a hard task because the set of all solutions of (1) is very difficult to characterize for a general matrix A. It can be viewed as being the union of the set of solutions commuting with A with the set of solutions non commuting with A. Some results characterizing the commuting solutions for particular choices of matrices A have been stated by Ding and Rhee and their collaborators. See, for instance, [4] for stochastic matrices and [6,7] for diagonalizable matrices; see also the references therein for other cases of A. Along with those characterizations, some methods for finding commuting solutions have also been proposed. For example, in [4], a method based on Brouwer's fixed point theorem was used for solving (1), under the assumption of A being stochastic; in [5], spectral based methods are proposed, and, in [6], an iterative scheme based on the mean ergodic theorem is derived for a diagonalizable matrix A. All of these papers contain many valuable contributions for understanding the YB-like equation and for computing its solutions. However, their implementation in finite precision environments has not been carried out and an investigation

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from a numerical viewpoint is lacking. Moreover, important features of those numerical methods like accuracy, stability and computational cost have not been investigated so far.

Hence, one of the goals of this work is to give a contribution to fill in this gap. We propose two iterative schemes for calculating commuting solutions of YB-like equation (1), regardless of A being or not diagonalizable, and provide a thorough investigation of their numerical behavior. Topics like convergence, stability, relative error, choice of a suitable initial guess, etc., will be investigated in detail. Both iterations are then implemented and compared with the iteration proposed in [6] for diagonalizable matrices A. Our methods are also tested with non-diagonalizable matrices. All the results of the implementations will be discussed. It will become clear that iteration (2) shows a superior numerical performance. Towards our aim, we present our first iterative scheme in what follows:

$$X_{k+1} = X_k^2(2X_k - I)^{-1}, \tag{2}$$

where X_0 is a certain starting matrix and $k = 0, 1, 2, \ldots$. Assuming that X_0 has no eigenvalue with real part equal to 1/2 and commutates with A, it is proven in Section 2 that the sequence (X_k) generated by (2) is well defined and converges quadratically to an idempotent matrix X_* commuting with A, that is, $X_*^2 = X_*$ and $AX_* = X_*A$. Once such an X_* has been computed, it is immediate that

$$X = AX_*$$

is a solution of YB-like equation (1).

A complementary iterative scheme for evaluating the solutions of (1) is provided below.

Given an initial matrix Y_0 commuting with A and such that, for a given subordinate matrix norm,

$$\|AY_0A - Y_0AY_0\| < 1, \tag{3}$$

we will show that the sequence (Y_k) defined by

$$\begin{aligned} \psi_k &= (Y_k(Y_k - A))^2, \\ Y_{k+1} &= \frac{1}{2}\left(A + (A(A - 4\psi_k))^{1/2}\right), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{4}$$

converges quadratically to a solution of (1). If a matrix $Z \in \mathbb{C}^{n\times n}$ has no eigenvalues on the closed negative real axis, $Z^{1/2}$ stands for the principal matrix square root of Z, that is, $Z^{1/2}$ is the unique square root of Z with eigenvalues having positive real parts [9,10]. For the iteration (4) to be well defined, it shall be assumed that, during the iterative process, the eigenvalues of the matrix $A(A - 4\psi_k)$ do not lie on the closed negative real axis. Choosing the minus sign in the definition of Y_{k+1} instead of the plus one, yields a variant of (4)

$$\begin{aligned} \psi_k &= (Y_k(Y_k - A))^2, \\ Y_{k+1} &= \frac{1}{2}\left(A - (A(A - 4\psi_k))^{1/2}\right), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{5}$$

which converges to a different solution of (1).

The paper is organized as follows. Section 2 is devoted to the investigation of iteration (2). We show, in particular, that it converges quadratically to an idempotent matrix in a stable fashion, with respect to perturbations of first-order. Then a commuting solution of (1) comes out easily. The second iteration (4) is the topic of Section 3. It is proven that its convergence is quadratic in terms of a residual arising naturally from (1). Termination criteria and error estimation are addressed in Sections 4 and 5, respectively. To summarize the previous results, two practical algorithms are written in pseudo-code in Section 6. Both algorithms are then implemented and tested with several numerical experiments in Section 7. Finally, some conclusions of our work will be drawn in Section 8.

Notation: $\|.\|$ denotes a subordinate matrix norm and $\|.\|_F$ the Frobenius norm; $\sigma(A)$ denotes the spectrum of the matrix A; $\rho(A)$ is the spectral radius of A; $\Re(z)$ is the real part of the complex number z.

2. Analysis of iteration (2)

2.1. Convergence

To investigate the convergence of (2), we start by considering the scalar polynomial $p(z) = z^2 - z$, which has 0 and 1 as roots. Applying Newton's method to the equation $p(z) = 0$, yields the complex scalar iteration

$$z_{k+1} = \frac{z_k^2}{2z_k - 1}, \tag{6}$$

$(k = 1, 2, \ldots)$ which is the scalar counterpart of (2). For $\alpha = 0,\ 1,$ let

$$B(\alpha) := \{z_0 \in \mathbb{C} :\ z_k \to \alpha\},$$

denote the basin of attraction of the root α.