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Iterative techniques for the initial value problem for Caputo fractional differential equations with non-instantaneous impulses

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ABSTRACT

Two types of algorithms for constructing monotone successive approximations for solutions to initial value problems for a scalar nonlinear Caputo fractional differential equation with non-instantaneous impulses are given. The impulses start abruptly at some points and their action continue on given finite intervals. Both algorithms are based on the application of lower and upper solutions to the problem. The first one is a generalization of the monotone iterative technique and it requires an application of the Mittag-Leffler function with one and two parameters. The second one is easier from a practical point of view and is applicable when the right hand sides of the equation are monotone. We prove that the functional sequences are convergent and their limits are minimal and maximal solutions of the problem. An example is given to illustrate the results.

1. Introduction

In the real world life there are many processes and phenomena that are characterized by rapid changes in their state. In the literature there are two popular types of impulses:

- *instantaneous impulses*- the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, for example, [7,13], the monographs [12,17,24] and the cited references therein);
- *non-instantaneous impulses* - an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. Hernandez and O'Regan [11] introduced this new class of abstract differential equations where the impulses are not instantaneous and they investigated the existence of mild and classical solutions. For recent work we refer the reader to [8,19–21,25–27].

In this paper the impulses start abruptly at some points and their action continue on given finite intervals. As a motivation for the study of these systems we consider the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some

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intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval. The model of this situation is the so called non-instantaneous impulsive differential equation.

This paper considers an initial value problem for a nonlinear scalar non-instantaneous impulsive Caputo fractional differential equation on a closed interval. The monotone iterative technique combined with the method of lower and upper solutions is applied to find approximately the solution of the given problem. A procedure for constructing two monotone functional sequences is given. The elements of these sequences are solutions of suitably chosen initial value problems for scalar linear non-instantaneous impulsive differential equations for which there is an explicit formula. Also, the elements of these sequences are lower/upper solutions of the problem. We prove that both sequences converge and their limits are minimal and maximal solutions of the problem. The Mittag-Leffler function with one and two parameters is included in the explicit formula for the successive approximations. This makes the practical application of the suggested algorithm harder. In the partial case of monotonic right side parts of the studied equation another iterative scheme for approximate solving is applied. This method is based also on the application of lower and upper solutions. An example is given to illustrate the procedure.

We note that iterative techniques combined with lower and upper solutions are applied in the literature to approximately solve various problems for ordinary differential equations [16], for second order periodic boundary value problems [5], for differential equations with maxima [1,9], for difference equations with maxima [2], for impulsive differential equations [4,7], for impulsive integro-differential equations [10], for impulsive differential equations with supremum [13], for differential equations of mixed type [14], for Riemann–Liouville fractional differential equations [3,18,22], and for Caputo impulsive fractional differential equations [6].

2. Non-instantaneous impulses in Caputo fractional differential equations

In this paper we will assume two increasing finite sequences of points $\{t_i\}_{i=0}^{p+1}$ and $\{s_i\}_{i=0}^{p+1}$ are given such that $t_0 = 0 < s_i < t_{i+1} < s_{i+1}$, $i = 0, 1, 2, \dots, p$, and $T = s_{p+1}$, p is a natural number.

Consider the initial value problem (IVP) for the nonlinear *non-instantaneous impulsive fractional differential equation* (NIFrDE)

$$\begin{aligned} &{}_0^cD^q x(t) = f(t,x) \text{ for } t \in (t_k, s_k],\ \ k = 0, 1, \dots, p, p+1,\\ &\quad x(t) = \phi_k(t, x(t), x(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\ \ k = 0, 1, 2, \dots, p,\\ &\quad x(0) = x_0, \end{aligned} \tag{1}$$

where $x, x_0 \in \mathbb{R},\ f : \cup_{k=0}^{p+1}[t_k, s_k] \times \mathbb{R} \to \mathbb{R},\ \phi_k : [s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (k = 0, 1, 2, \dots, p)$.

Also consider the corresponding IVP for FrDE

$${}_\tau^cD^q x(t) = f(t,x) \quad \text{for } t \in\ [\tau, s_k] \quad \text{with} \quad x(\tau) = \tilde{x}_0, \tag{2}$$

where $\tau \in [t_k,\ s_k)$, $k = 0, 1, 2, \dots, p+1$.

The IVP for FrDE (2) is equivalent to the following Volterra integral equation

$$x(t) = \tilde{x}_0 + \frac{1}{\Gamma(q)} \int_\tau^t (t-s)^{q-1} f(s, x(s)) ds \quad \text{for } t \in\ [\tau, s_k]. \tag{3}$$

In the special case of a linear fractional differential equation an explicit formula for the solution is obtained (see, for example, [15]),i.e. if we consider the IVP for the linear FrDE

$${}_\tau^cD^q u(t) - \lambda u(t) = h(t) \quad \text{for } t \in\ [\tau, b] \quad \text{with} \quad u(\tau) = a, \tag{4}$$

then according to Section 4.3.1 [15] the solution is given by

$$u(t) = aE_q(\lambda(t-\tau)^q) + \int_\tau^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) h(s) ds. \tag{5}$$

We introduce the following classes of functions

$$\begin{aligned} NPC^1 = \{&u : [0, T] \to \mathbb{R} :\ u \in C^1(\cup_{k=0}^{p+1}[t_k, s_k], \mathbb{R}) :\\ &u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty,\ \ u'(s_k) = \lim_{t \uparrow s_k} u'(t) < \infty,\ \ k = 0, 1, 2, \dots, p+1\},\\ &u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty,\ \ k = 0, 1, 2, \dots, p+1\},\\ PC^1([0, T]) = \{&u : [0, T] \to \mathbb{R} :\ u \in NPC^1,\ \ u \in C(\cup_{k=0}^{p+1}(s_k, t_{k+1}]\,\mathbb{R})\}. \end{aligned}$$

Definition 1. A function $x \in PC^1([0, T], \mathbb{R}^n)$ is called a solution of the IVP for NIFrDE (1) if

- $x(t) = x_k(t)$ for $t \in (t_k, s_k], k = 0, 1, 2, \dots, p+1,$ where $x_k \in C([t_k, s_k], \mathbb{R}^n),$ satisfies ${}_{t_k}^cD^q x_k(t) = f(t, x_k(t))$ a.e. on $(t_k,\ s_k)$ with $x_k(t_k) = x_0$ if $k = 0$ and $x_k(t_k) = \tilde{x}_k(t_k)$ if $k = 1, 2, \dots, p+1$;