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Richardson extrapolation technique for singularly perturbed system of parabolic partial differential equations with exponential boundary layers

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ABSTRACT

In this article, we propose a higher-order uniformly convergent numerical scheme for singularly perturbed system of parabolic convection-diffusion problems exhibiting overlapping exponential boundary layers. It is well-known that the the numerical scheme consists of the backward-Euler method for the time derivative on uniform mesh and the classical upwind scheme for the spatial derivatives on a piecewise-uniform Shishkin mesh converges uniformly with almost first-order in both space ant time. Richardson extrapolation technique improves the accuracy of the above mentioned scheme from first-order to second-order uniformly convergent in both time and space. This has been proved mathematically in this article. In order to validate the theoretical results, we carried out some numerical experiments.

1. Introduction

Singular perturbation problem (SPP) is an important class of initial-boundary-value problems which have broaden applications. The well-known examples of system of singularly perturbed parabolic convection-diffusion problems are the diffusion-convection enzyme model, tubular model in chemical reactor theory and neutron transport problem, where the diffusion coefficients are taken sufficiently small to have layer phenomena. Further details can be found in the book of Pao [18] for problems modeled by convection-diffusion equations. When the perturbation parameter is sufficiently small, the solution of the problem will have significantly large (partial) derivatives near the boundary, which illustrates the boundary layers. The existence of boundary layers brings difficulty to numerical solutions of the model problem. Also, it makes the standard numerical methods unstable and fail to yield accurate results. In order to overcome this difficulty, one has to search for some special schemes on uniform or nonuniform meshes. This phenomenon leads to develop the concept of ε-uniformly convergent numerical methods; in which the order of convergence and the error constant are independent of the singular perturbation parameter ε. Various ε-uniform numerical schemes are developed in the literature for singular perturbation problems, for more details one can refer the books [5,6,21].

Numerical methods for singularly perturbed system of ordinary differential equations (ODEs) have been studied extensively by many authors. To cite a few: Bellew and O'Riordan [1] examined an upwind difference scheme to solve coupled system of convection-diffusion equations. Cen [2] considered the finite difference method on Shishkin mesh to solve weakly

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coupled system of convection-diffusion problems. In both the cases, the analysis is being performed on the piecewise-uniform Shishkin mesh and it is shown that the accuracy of the method is of $O(N^{-1}\ln^2 N)$ and $O(N^{-1}\ln N)$, respectively. O'Riordan and Stynes [17] analyzed an upwind difference scheme for a strongly coupled system of singularly perturbed convection-diffusion problems. Later, Liu and Chen [13] proposed an ε-uniformly convergent scheme for singularly perturbed system of ODE, by using the mesh equidistribution technique. However, the theory and the numerical solution of singularly perturbed system of partial differential equation (PDE) are still at the primary stage. To obtain the uniform convergent numerical solution for the singularly perturbed system of PDE of reaction-diffusion type, classical finite difference schemes are applied on the Shishkin mesh by Gracia and Lisbona [8]. Gracia et al. [9] applied the central difference scheme to solve the coupled system of singularly perturbed parabolic reaction-diffusion equations. Also Franklin et al. [7] studied an upwind finite difference scheme for singularly perturbed system of linear parabolic reaction-diffusion problems. Recently, Rao and Srivastava [19] applied combination of HODIE scheme and the central difference scheme in space with backward Euler method in time direction to solve time-dependent singularly perturbed system of convection-diffusion problems of one parameter family.

Richardson extrapolation, a well-known post-processing technique, which provides a better accuracy to the exact solution obtained by averaging the numerical solutions computed on two embedded meshes. One can find several articles in the literature, where Richardson extrapolation technique has been used to solve SPPs. To cite a few: Natividad and Stynes [16] analyzed this technique for singularly perturbed convection-diffusion boundary-value problem (BVP). Mohapatra and Natesan [14] applied this technique for singularly perturbed delay BVP. This approach has been established for singularly perturbed coupled system of convection-diffusion BVP by Deb and Natesan in [4]. Extrapolation technique was adapted to solve singularly perturbed parabolic PDE by Mukherjee and Natesan [15]. Shishkin and Shishkina [22] proposed a Richardson extrapolation technique for singularly perturbed reaction-diffusion problem. Recently, Das and Natesan [3] analyzed the Richardson extrapolation technique for solving singularly perturbed delay parabolic PDE. However, no theoretical framework for analyzing Richardson extrapolation technique for singularly perturbed system of parabolic PDE is available in the literature.

The goal of this paper is to construct and analyze the Richardson extrapolation technique to obtain better numerical approximate solution to the model problem (2.1). In the proposed numerical scheme the time derivative is approximated by the backward-Euler scheme on uniform mesh and the spatial derivatives are approximated by the classical upwind finite difference scheme on the piecewise-uniform Shishkin meshes. We show that this implicit upwind scheme is of first-order uniformly convergent. Then we implement the Richardson extrapolation technique that improves the accuracy of the simple implicit upwind finite difference scheme from almost first-order to second-order uniformly convergent. This has been proved theoretically by obtaining almost second-order uniformly convergent error estimates. Numerical experiments are carried out to show the accuracy and efficiency of the proposed method.

We organize the rest of the paper as follows: In Section 2, we establish the continuous problem. Further, we derive the bounds for the smooth and singular components. Section 3 describes the piecewise-uniform Shishkin mesh and the classical implicit upwind difference scheme for discretization of the continuous problem. In Section 4, the Richardson extrapolation technique has been described. Section 5 contains numerical experiments to confirm the theoretical results. We end this article with the summary.

Notation: As usual, throughout this paper C (sometimes subscripted) denotes a generic positive constant, which is independent of ε_1, ε_2 and of mesh, it may take different values at different places. Note that an unsubscripted C may take different values in different places, but a subscripted C is a fixed constant that does not change throughout the paper. Here, we are interested in maximum-norm error estimates.

On each closed set $\overline{Q} = \overline{\Omega} \times [0, T]$, we define the maximum-norm for each real-valued function $\psi \in \overline{Q}$ by $\|\psi\|_{\overline{Q}} = \max_{(x,t)\in\overline{Q}} |\psi(x,t)|$. For vector-valued function $\vec{u}(x,t) := (u_1(x,t), u_2(x,t))$, set $|\vec{u}(x,t)| = (|u_1(x,t)|, |u_2(x,t)|)^T$, and we define the maximum norm as

$$\|\vec{u}\|_{\overline{Q}} = \max_{1\le i\le m}\left\{\|u_1(x,t)\|_{\overline{Q}}, \|u_2(x,t)\|_{\overline{Q}}\right\}.$$

On $\overline{Q}^{N,M} := \{x_i, t_n\}_{i,n=0}^{N,M}$, we define discrete maximum norm for real-valued mesh function $U := \{U(x_i, t_n)\}_{i,n=0}^{N,M}$ by $\|U\|_{\overline{Q}^{N,M}} = \max_{(x_i,t_n)\in Q^{N,M}} |U(x_i, t_n)|$, for vector-valued mesh function

$\vec{U}(x_i, t_n) := (U_1(x_i, t_n), U_2(x_i, t_n))^T$, set $|\vec{U}(x_i, t_n)| = (|U_1(x_i, t_n)|, |U_2(x_i, t_n)|)^T$ and discrete maximum norm is defined by $\|\vec{U}\|_{\overline{Q}^{N,M}} = \max\left\{\|U_1\|_{\overline{Q}^{N,M}}, \|U_2\|_{\overline{Q}^{N,M}}\right\}$.

2. The continuous problem

In this article, we consider singularly perturbed system of parabolic convection-diffusion initial-boundary-value problem (IBVP) on the domain $Q := \Omega \times (0, T], \Omega = (0, 1)$: