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A study of a covering dimension of finite lattices

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ABSTRACT

Indubitably, the notion of covering dimension of frames was, extensively, studied. Many searches such as Charalambous, Banashewski and Gilmour (see, for example (Charalambous, 1974; Charalambous, 1974 [11]; Banaschewski and Gilmour, 1989 [12]) studied this dimension. Also, in the study [5], the covering dimension of finite lattices has been characterized by using the so called minimal covers. This approach gave the motive to other searches such as Zhang et al., to study properties of this dimension (see Zhang et al. (2017) [9]). In this paper, we study the covering dimension of finite lattices in combination with matrix theory. Essentially, we characterize the minimal covers of finite lattices and the order of those covers using matrices and we compute the covering dimension of the corresponding finite lattices.

1. Introduction

In the study [5], adapting the usual definition of the topological covering dimension (see, for example [4]), the meaning of the covering dimension, $\dim(X)$, of a bounded lattice X and the meaning of the order, $\mathrm{ord}(C)$, of a subset C of X were studied in the class of finite lattices based on the study [10], where the covering dimension in the class of frames was studied. Many properties of this dimension were studied, mentioning the characterization of the covering dimension of finite lattices by their minimal covers:

Theorem 1.1. *Let X be a finite lattice and $k \in \{0, 1, 2, \ldots\}$. Then $\dim(X) \leqslant; k$ if and only if every minimal cover C of X satisfies* $\mathrm{ord}(C) \leqslant k$.

Corollary 1.2. *Let X be a finite bounded lattice and*

$$k = \max\{ord(C) : C \textit{ is a minimal cover of } X\}.$$

Then $\dim(X) = k$.

Also, in the study [5], it was observed and proved that each minimal cover of a finite lattice is an antichain and open problems were posted, some of which were answered in the study [9].

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In studies [6,7], the classical topological covering dimension was characterized through the matrix theory, and in study [8], the order dimension of a finite poset was studied by using matrices and the notion of the order-matrix. In this study we are going to characterize the covering dimension of an arbitrary finite lattice with matrices.

In particular, the rest of the paper is organized as follows: In Section 2, we give basic definitions and notations which will be used in this study. In Section 3, we characterize the minimal covers of finite lattices using order-matrices. In Section 4, we characterize the meaning of the order of those covers and, consequently, the covering dimension of finite lattices through the matrix theory. Finally, in Sections 5 and 6, we present algorithms for computing the covering dimension of an arbitrary finite lattice.

2. Preliminaries

Our references are [1–3,5]. In what follows, we use the notation of a finite lattice $(X, \leqslant)$ writing $X = \{x_1, x_2, \ldots, x_n\}$, where $x_1 = 0_X$ and $x_n = 1_X$ are the bottom and the top elements of the lattice X, respectively.

Also, if $C \subseteq X$, then by $\bigvee C$ and $\bigwedge C$, we denote the supremum and the infimum of C, respectively and by $\mathcal{UB}(C)$, we denote the set of all upper bounds of C.

Two elements x_i and x_j of X are said to be *comparable* if $x_i \leqslant x_j$ or $x_j \leqslant x_i$. Otherwise they are said to be *incomparable* (we write $x_i \| x_j$). A subset of X in which every pair of elements is comparable is called a *chain*. A subset of X in which every two distinct elements are incomparable is called an *antichain*. If A is an antichain, then

$$Pl(A) = \{x \in X : x \parallel a \text{ for every } a \in A\}.$$

For every $x_i \in X$ we set $\downarrow^* x_i = \{x \in X : x \leqslant x_i\} \setminus \{0_X\}$.

Definition 2.1. We say that a cover of the lattice X is a subset C of X such that $0_X \notin C$ and $\bigvee C = 1_X$. A cover D of X is called *refinement* of a cover C of X, written $D \ll C$, if for every $d \in D$, there exists $c \in C$ such that $d \leqslant c$.

We recall that the relation $\ll$ is quasiorder (or preorder) (reflexive and transitive) but not in general antisymmetric.

We also use the following notation:

$$\mathbb{N}_0 = \{0, 1, 2, \ldots\} \quad \text{and} \quad \overline{\mathbb{N}} = \mathbb{N}_0 \cup \{\infty\}.$$

Definition 2.2. Let X be a bounded lattice and $k \in \mathbb{N}_0$. The *order* of a subset C of X is defined as follows:

(a) $ord(C) = k$, if the infimum of any $k+2$ distinct elements of C is 0_X and there exist $k+1$ distinct elements of C whose infimum is not 0_X.
(b) $ord(C) = \infty$, if for every $k \in \mathbb{N}_0$, there exist k distinct elements of C whose infimum is not 0_X.

Definition 2.3. Let **BLat** be the class of bounded lattices. We define the *covering dimension function*

$$\dim : \mathbf{BLat} \longrightarrow \mathbb{N}_0$$

as follows:

(1) $\dim(X) \leqslant k$, if every finite cover C of X has a finite refinement R with $\mathrm{ord}(R) \leqslant k$.
(2) $\dim(X) = k$, where $k \geq 1$, if $\dim(X) \leqslant k$ and $\dim(X) \nleq k-1$.
(3) $\dim(X) = \infty$, if the inequality $\dim(X) \leqslant k$ does not hold for any $k \in \mathbb{N}_0$.

Definition 2.4. A cover C of X is called *minimal* if $C \subseteq R$ for every refinement R of C.

We mention that a minimal cover is the bottom element of the poset $(Cov(X), \subseteq)$, where $Cov(X)$ is the set of all covers of X. Clearly, the cover $\{x_n\}$ is minimal if and only if every cover of X contains the top element x_n.

Definition 2.5. Let $(X, \leqslant)$ be a finite poset, where $X = \{x_1, x_2, \ldots, x_n\}$. The $n \times n$ matrix $T_X^{\leqslant} = (t_{ij})$, where

$$t_{ij} = \begin{cases} 1, & \text{if } x_i \leqslant x_j \\ 0, & \text{otherwise} \end{cases}$$

is called the *incidence matrix* of X.

We denote by $c_1(T_X^{\leqslant}), \ldots, c_n(T_X^{\leqslant})$ the n columns of the incidence matrix $T_X^{\leqslant}$.

Definition 2.6. Let $(X, \leqslant)$ be a finite poset, where $X = \{x_1, x_2, \ldots, x_n\}$. The $n \times n$ matrix $\mathbf{A}_X^{\leqslant} = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & if\ i = j \\ 2, & if\ x_i < x_j \\ -2, & \text{if } x_j < x_i \\ 0, & \text{if } x_i \parallel x_j \end{cases}$$

is called the *order-matrix* of X.