Applied Mathematics and Computation 333 (2018) 362–368

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Sufficient conditions for hypergraphs to be maximally edge-connected

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ARTICLE INFO

Keywords:
Edge-connectivity
Hypergraph
Maximally edge-connected

ABSTRACT

Let H be a connected hypergraph. H is said to be linear if any two edges of H share at most one vertex. If all edges of H have the same cardinality, then H is uniform. We call H maximally edge-connected if the edge-connectivity of H attains its minimum degree. In this paper, we present some sufficient conditions for linear uniform hypergraphs to be maximally edge-connected that generalize the corresponding well-known results for graphs.

1. Introduction

One of the classical reliable measures for a graph G is the edge-connectivity $\lambda(G)$, which is defined as the minimum number of edges whose removal leaves G disconnected. If $\delta(G)$ is the minimum degree, then it is well-known that $\lambda(G) \le \delta(G)$ [12]. It is therefore interested in studying graphs satisfying $\lambda(G) = \delta(G)$ (so called maximally edge-connected). Known results are referred to [5] and a survey by Hellwing and Volkmann [7].

Hypergraphs, a generalization of graphs, have been widely and deeply studied in [1,2], and quite often have proved to be a successful tool to represent and model concepts and structures in various areas of computer science and discrete mathematics. As a way to deal with particular problems arising in computer science, hypergraphs are often used to model networks which provide a communication link between two or more processors. And there exists one basic parameter that is generally considered for hypergraphs in relation to the fault-tolerance of networks: the edge-connectivity.

A *hypergraph* $H = (V, E)$ is an ordered pair, where V is the set of *vertices* and E is the set of *edges* of H, which are subsets of V. Unless specified otherwise, we consider only simple hypergraphs, i.e., hypergraphs whose edges are distinct. A hypergraph is *r-uniform* if each edge is an r-set (a set of r-elements). Thus a 2-uniform hypergraph is a graph. If any two distinct edges of H share at most one vertex, then H is *linear*. A vertex $v \in V$ is said to be *incident* with an edge $e \in E$, if e is a subset of V containing v. Two distinct vertices of H are *adjacent* or *neighbors* if some edge of H contains both. The *neighborhood* $N(u)$ of a vertex u is the set of vertices of H that are adjacent to u. The *degree* of a vertex v, denoted by $d(v)$, is the number of edges in which v is contained. Similar to graphs, we can also define the *minimum degree* $\delta(H)$ and the *maximum degree* $\Delta(H)$. A *walk* W in a hypergraph H is a finite alternating sequence $W = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$, where v_i is a vertex for $i = 0, 1, \ldots, k$ and e_j is an edge such that $v_{j-1}, v_j \in e_j$ for $j = 1, 2, \ldots, k$. A walk W is a *path* if all the vertices v_i for $i = 0, 1, \ldots, k$ and all the edges e_j for $j = 1, 2, \ldots, k$ in W are distinct. The value k is the *length* of the path, and the *distance* between two vertices v_0 and v_k, denoted by $d(v_0, v_k)$, is defined as the length of a shortest path between v_0 and v_k. A hypergraph is *connected*, if there is a walk between any two of its vertices, otherwise it is *disconnected*. The *diameter* of H *diam*(H), is the greatest distance between any pair of vertices of H, if H is connected.

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https://doi.org/10.1016/j.amc.2018.03.109
0096-3003/© 2018 Published by Elsevier Inc.

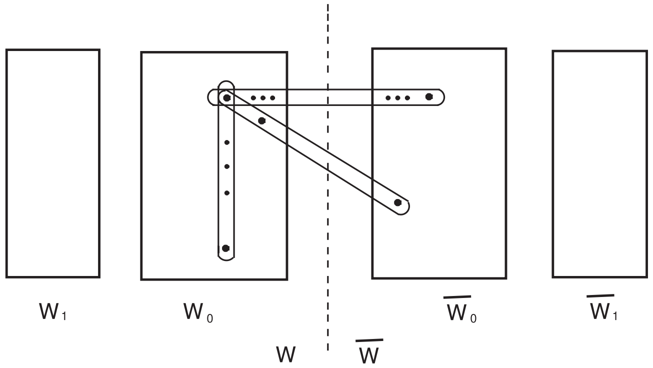

Fig. 1. The proof of Theorem 2.2.

The above definitions of edge-connectivities of graphs and maximally edge-connected graphs can extend in a natural way to hypergraphs. If S is a subset of $E(H)$, then $H-S$ denotes the hypergraph $(V, E-S)$. We say that S is an *edge-cut*, if $H-S$ is disconnected. The *edge-connectivity* $\lambda(H)$ of a connected hypergraph H is the minimum cardinality among all the edge-cuts of H.

The edge-connectivities of hypergraphs have been investigated in several recent papers. For example, Zykov [14] gave a Menger-type theorem for hypergraphs. The authors in [3,4] studied edge augmentation of hypergraphs. Gu and Lai in [13] presented necessary and sufficient conditions for an r-uniform hypergraphic sequence to have a k-edge-connected realization. In [6], Dankelmann and Meierling generalized some well-known sufficient conditions for maximally edge-connected graphs to hypergraphs.

Proposition 1.1. *[6]. Let H be a hypergraph. Then* $\lambda(H) \leq \delta(H)$.

Then we say that H is *maximally edge-connected*, if $\lambda(H) = \delta(H)$. The following result is a generalization of the classical result by Plesník [9] for linear uniform hypergraphs.

Theorem 1.2. *[6]. Let H be a linear uniform hypergraph with* $diam(H) \leq 2$. *Then* $\lambda(H) = \delta(H)$.

The aim of this paper is to investigate how some well-known sufficient conditions for graphs to be maximally edge-connected can be generalized to hypergraphs. In Section 2, first we will give a new distance sufficient condition for linear uniform hypergraphs to be maximally edge-connected. Besides, we present examples to illustrate that some distance conditions for maximally edge-connected bipartite graphs can not be generalized to linear uniform bipartite hypergraphs. In Section 3, we confirm a sufficient condition in terms of diameter and girth that ensures linear uniform hypergraphs to be maximally edge-connected.

2. A new distance condition

In [9], Plesník showed that any graph G with $diam(G) \leq 2$ is maximally edge-connected. In 1989, Plesník and Znám [10] relaxed the condition in the sense that some distances can be greater than 2.

Theorem 2.1. *[10]. If in a connected graph G no four vertices* u_1, v_1, u_2, v_2 *with*

$$d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2) \geq 3$$

exist, then $\lambda(H) = \delta(H)$.

Below we show that the above result holds for linear r-uniform hypergraphs with $r \geq 3$.

Theorem 2.2. *Let H be a linear r-uniform hypergraph with* $r \geq 3$. *If there exist no four vertices* u_1, v_1, u_2, v_2 *with*

$$d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2) \geq 3$$

then $\lambda(H) = \delta(H)$.

Proof. Let $H=(V, E)$ and let $S \subseteq E$ be a minimum edge-cut of H. Then let W and $\overline{W}$ be the vertex sets of two parts arising after deleting S from H. Further, let $W_0 \subseteq W$ and $\overline{W_0} \subseteq \overline{W}$ be the set of vertices incident with edges of S and put $W_1 := W - W_0$, $\overline{W_1} := \overline{W} - \overline{W_0}$ (see Fig. 1). Denote the cardinalities of $W_1, W_0, \overline{W_0}$ and $\overline{W_1}$ by $a_1, a_0, \bar{a}_0$, and $\bar{a}_1$.

The distance condition in our theorem implies that $a_1 \leq 1$ or $\overline{a_1} \leq 1$. Without loss of generality, we may assume that $a_1 \leq 1$. Consider the following two cases.

Case 1: $a_1 = 0$

Then each vertex in W is incident with some edges of S.