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Partitioning the Cartesian product of a tree and a cycle☆

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ABSTRACT

Let $G = (V, E)$ be a graph of order n, and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ a sequence of positive integers. The sequence λ is admissible for G if $\lambda_1 + \cdots + \lambda_p = n$. Such an admissible sequence λ is said to be realizable in G if there exists a partition $(V_1, V_2, \dots, V_p)$ of the vertex set V such that V_i induces a connected subgraph of order n_i in G for each i. If every admissible sequence is realizable in G, then we say that G is arbitrarily partitionable (AP, for short). We show that if a tree T of maximum degree at most $n+1$ has a path containing all the vertices of degree $n+1$, then $T \square C_n$ has a Hamiltonian path. In particular, for any caterpillar T with maximum degree at most $n+1$, $T \square C_n$ is AP. In addition, if T is a caterpillar with $\Delta(T) \geq n+4$, then $T \square C_n$ is not AP. For the cases $n+2 \leq \Delta(T) \leq n+3$, we present some sufficient conditions for a caterpillar T such that $T \square C_n$ is AP.

1. Introduction

Let $G = (V, E)$ be a simple graph of order n, and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ a sequence of positive integers. The sequence λ is *admissible* for G if $\lambda_1 + \cdots + \lambda_p = n$. Such an admissible sequence λ is said to be *realizable* in G if there exists a partition $(V_1, V_2, \dots, V_p)$ of the vertex set V such that each V_i induces a connected subgraph of order n_i in G. If every admissible sequence is realizable in G, then we say that G is *arbitrarily partitionable* (AP, for short). For a given integer $k \in \{1, 2, \dots, n\}$, a graph G is called *k-AP* if every admissible sequence λ of G of length at most k is realizable in G. The notion of AP graphs was first introduced by Barth et al. [1], and independently, by Horňák and Woźniak [17]. Observe that having a (almost) perfect matching is necessary for a graph to be AP. So, the independence number of an n-vertex AP graph is at most $\lceil \frac{n}{2} \rceil$. Questions related to arbitrarily partitions of graphs are old. In 1978 and 1977 respectively, Győri [14] and Lovász [23] independently proved that k-connected graphs are k-AP.

It is clear that a connected graph is AP if one of its spanning trees is AP. The problem of deciding whether a given admissible sequence is realizable in a given graph G is NP-complete [1]. We refer to Refs. [2,11,12] for algorithmic aspect of AP graphs. Various properties on AP trees have been shown. It is not known whether it is NP-complete for the class of all trees. In [1], Barth et al. showed that this problem is polynomial for the class of tripodes. In [17], Horňák and Woźniak showed that the maximum degree of a AP tree is at most 6. Later, this bound has been decreased to 4 in [2]. Cichacz et al. [13] gave a complete characterization of AP caterpillars with four leaves. They also described two infinite families of AP trees with maximum degree three or four. Ravaux [26] focused on trees with a large diameter. There are also some results on AP star-like trees, unicyclic AP graphs and the shape of AP trees [3,18,21].

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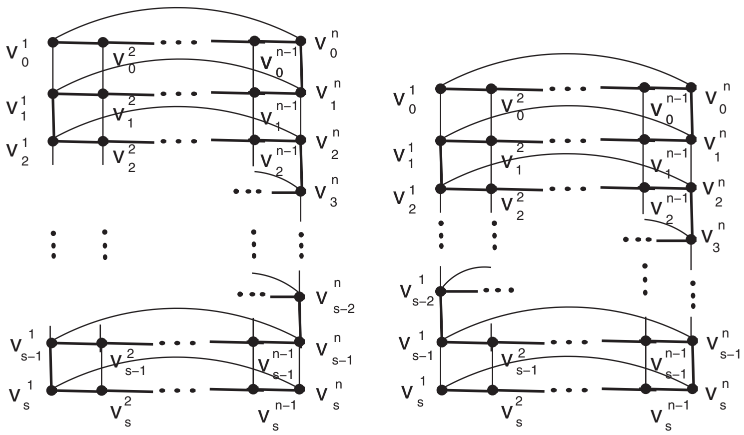

Fig. 1. A Hamiltonian path P_0 of $T_0\square C_n$.

A graph is said to be *traceable* if it contains a Hamiltonian path. It is clear that each traceable graph is AP. Therefore, a condition for a graph being traceable implies that the graph is AP. So, there are some relevant works [15,20,24,25] trying to replace some known conditions for traceability by the weaker ones implying that a graph is AP.

There are other families of AP graphs, such as OL-AP, R-AP and AP+k. For results on these aspect, we refer to [5–7,9,16,19,22]. Some structure of AP graphs and minimal AP graphs are investigated in [8,10].

Given two graphs G and H, the *Cartesian product* of G and H, denoted by $G\square H$, is the graph whose vertex set is $V(G)\times V(H)$ and two vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G\square H$ if and only if either $v_1 = v_2$ and $u_1u_2 \in E(H)$ or $u_1 = u_2$ and $v_1v_2 \in E(G)$. In [6], Baudon et al. conjectured that the Cartesian product of an AP graph and a traceable graph is AP and proved that the conjecture holds if the order of the traceable graph is at most four.

For a vertex v of a graph G, let $d(v)$ denote the degree of v, and $\Delta(G)$ is the maximum degree of G. In this paper, we focus on the Cartesian product of a tree and a cycle. In Section 2, we show that if a tree T with $\Delta(T) \le n+1$ has a path containing all the vertices of degree $n+1$, then $T\square C_n$ is traceable. In particular, $T\square C_n$ is AP for any caterpillar T with $\Delta(T) \le n+1$. In Section 3, we show that for a caterpillar T, the Cartesian product $T\square C_n$ is not AP if $\Delta(T) \ge n+4$. For the cases $n+2 \le \Delta(T) \le n+3$, we present some sufficient conditions for a caterpillar T such that $T\square C_n$ is AP.

2. Cartesian product of a tree and a cycle

Let G be a graph. For a vertex v, let $N(v)$ denote the set of neighbors of v in G, and for a set $S \subseteq V(G)$, let $N(S) = \cup_{v\in S}N(v)$. The distance of vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest path joining u and v in G. In the sequel, we denote the vertices of the cycle C_n in the clockwise direction by $u_1, u_2, \ldots, u_n$ and the vertex (v, u_j) of $T\square C_n$ by v^j.

In 1982, Batagelj and Pisanski proved the following elegant theorem.

Theorem 2.1. *(Batagelj, Pisanski [4]) For an integer $n \ge 3$ and a tree T, the Cartesian product $T\square C_n$ is Hamiltonian if and only if $\Delta(T) \le n$.*

Corollary 2.2. *Let G be a connected graph, and H be a Hamiltonian graph of order n. If G has a spanning tree of maximum degree at most n, then $G\square H$ is AP.*

Theorem 2.3. *Let T be a tree with $\Delta(T) \le n+1$ for an integer $n \ge 3$. If a path of T contains all the vertices of degree $n+1$, then $T\square C_n$ is traceable.*

Proof. In view of Theorem 2.1, we assume that $\Delta(T) = n+1$. Let $v_0v_1, \ldots, v_s$ be a maximal path in T which contains all the vertices of maximum degree. Clearly, $d(v_0) = d(v_s) = 1$, $d(v_i) \le n$ for $v_i \in V(T)\backslash V_0$, where $V_0 = \{v_0, v_1, \ldots, v_s\}$. Let $t = \max\{d_T(v, u) : v \in V_0, u \in V(T) \setminus V_0\}$. Let $V_1 = N(V_0) \setminus V_0$ and $V_{i+1} = N(V_i) \setminus V_{i-1}$ for any $i \in \{1, \ldots, t-1\}$. Since T is a tree and $G[V_0]$ is connected, V_i is an independent set for each $i \ge 1$. Let $T_i = T[\cup_{j=0}^{i}V_j]$ for $i \in \{0, 1, \ldots, t\}$. Clearly, $T_t \cong T$, and $T_i\square C_n$ is a subgraph of $T_j\square C_n$ for $i \le j$. We construct a Hamiltonian path of $T\square C_n$ recursively.

Step 1: Take a Hamiltonian path P_0 (in bold lines) of $T_0\square C_n$ as shown in Fig. 1.

$$P_0 = \begin{cases} \underline{v_0^1v_0^2\ldots v_0^n}\ \underline{v_1^nv_1^{n-1}\ldots v_1^1}\ \underline{v_2^1v_2^2\ldots v_2^n}v_3^n\ldots v_{s-1}^1\underline{v_s^1v_s^2\ldots v_s^n}, & \text{if } s \text{ is even} \\ \underline{v_0^1v_0^2\ldots v_0^n}\ \underline{v_1^nv_1^{n-1}\ldots v_1^1}\ \underline{v_2^1v_2^2\ldots v_2^n}v_3^n\ldots v_{s-1}^n\underline{v_s^nv_s^{n-1}\ldots v_s^1}, & \text{if } s \text{ is odd.} \end{cases}$$

Step 2: Extend the Hamiltonian path P_0 of $T_0\square C_n$ to a Hamiltonian path P_1 of $T_1\square C_n$.