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On distances in vertex-weighted trees

Qingqiong Cai[a,1], Fuyuan Cao[b], Tao Li[a,2,*], Hua Wang[c,d,3,*]

[a] College of Computer and Control Engineering, Nankai University, Tianjin 300071, China
[b] Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, School of Computer and Information Technology, Shanxi University, Taiyuan, Shanxi 030006, China
[c] College of Software, Nankai University, Tianjin 300071, China
[d] Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, USA

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ABSTRACT

The study of extremal problems on various graph invariants has received great attention in recent years. Among the most well known graph invariants is the sum of distances between all pairs of vertices in a graph. This is also known as the Wiener index for its applications in Chemical Graph Theory. Many interesting properties related to this concept have been established for extremal trees that maximize or minimize it. Recently a vertex-weighted analogue of sum of distances is introduced for vertex weighted trees. Some extremal results on (vertex-weighted) trees were obtained, by Goubko, for trees with a given degree sequence. In this note we first analyze the behavior of vertex-weighted distance sum in general, identifying the "middle part" of a tree analogous to that with respect to the regular distance sum. We then provide a simpler approach (than that of Goubko's) to obtain a stronger result regarding the extremal tree with a given degree sequence. Questions and directions for potential future study are also discussed.

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1. Introduction

An important problem in extremal graph theory deals with finding structures that maximize or minimize a particular real-valued graph invariant. For examples of some classic as well as recent work on graph invariants one may see [4,6,7,13,16,17,28] and the references therein. Among most well known graph invariants is the sum of distances (of a graph G)

$$\sigma(G) = \sum_{u,v\in V(G)} d_G(u,v) \tag{1}$$

where the summation covers all unordered pairs of vertices of G and $d_G(u,v)$ (or simply $d(u,v)$ when there is no confusion) is the distance between u and v in G.

* Corresponding authors at: Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, USA.
E-mail addresses: caiqingqiong@nankai.edu.cn (Q. Cai), cfy@sxu.edu.cn (F. Cao), litao@nankai.edu.cn (T. Li), hwang@georgiasouthern.edu (H. Wang).

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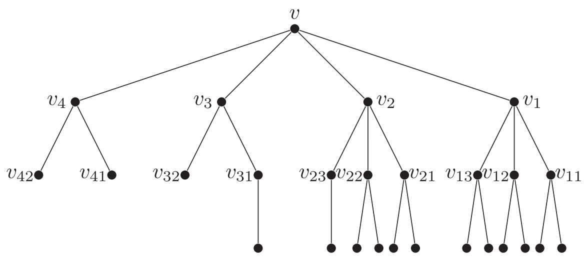

Fig. 1. A greedy tree with degree sequence $(4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 1, \ldots, 1)$.

The concept of $\sigma(\cdot)$ has been of interest to mathematicians for a long time. However, it received much more attention in the past two decades as the *Wiener index* of a graph, first introduced by the chemist Harry Wiener in 1947 [23–25]. Among other interesting observations, the properties of $\sigma(\cdot)$ in trees and the extremal trees that maximize or minimize it have been studied. For some of the most related such results and generalizations one may see [1–3,5,12,14,15,18–22,26,27,30,31] and the references therein. In particular, it was shown in [26,27] and [31] respectively that the *greedy tree* (Definition 1, also known as a Huffman tree in a more general sense) minimizes the value of $\sigma(\cdot)$ among trees with a given degree sequence (the non-increasing sequence of vertex degrees). In the same papers it was also mentioned that the *caterpillars* (trees whose removal of leaves results in a path) maximize the value of $\sigma(\cdot)$ among trees with a given degree sequence.

Definition 1 (Greedy trees)**.** With given vertex degrees, the greedy tree is constructed through the following "greedy algorithm":

(i) Label the vertex with the largest degree as v (the root);
(ii) Label the neighbors of v as $v_1, v_2, \ldots$, assign the largest available degrees to them such that $\deg(v_1) \geq \deg(v_2) \geq \cdots$;
(iii) Label the neighbors of v_1 (except v) as $v_{11}, v_{12}, \ldots$ such that they take all the largest available degrees and that $\deg(v_{11}) \geq \deg(v_{12}) \geq \cdots$, then do the same for $v_2, v_3, \ldots$;
(iv) Repeat (iii) for all the newly labeled vertices, always start with the neighbors of the labeled vertex with largest degree whose neighbors are not labeled yet.

For example, Fig. 1 shows such a greedy tree.

For a vertex-weighted graph G with weight function $w: V(G) \to \mathbb{R}^+$, each vertex $v \in V(G)$ has weight $w(v) > 0$. The *vertex-weighted Wiener index* for a vertex weighted graph is introduced in [11] as

$$\sum_{u,v \in V(G)} w(u)w(v)d(u,v) =: \sigma_W(G). \tag{2}$$

The notation *VWWI*$(\cdot)$ was used for this "vertex-weighted sum of distances". We will use $\sigma_W(\cdot)$ here to be consistent with (1).

Under the condition that the vertex weight is monotonic with respect to the vertex degree (i.e. the vertex with larger degree also has larger weight), the result of [26,27,31] was generalized to $\sigma_W(\cdot)$ in [8]. Most recently similar work on maximum $\sigma_W(\cdot)$ and caterpillars of a given degree sequence is presented in [9]. Although originally stated differently, in our terminologies the following was shown in [8].

Theorem 1.1 ([8])**.** *If weights are degree-monotone in vertex-weighted trees with a given degree sequence, the greedy trees minimize* $\sigma_W(\cdot)$.

The pioneering work in [8] involves defining many technical terms and lemmas. In this paper we will provide a simpler proof for a stronger result, that given the set of vertex weights and degree sequence, the *greedy vertex-weighted tree* (a greedy tree with the monotonic assignment of vertex weights with respect to the vertex degrees) minimizes $\sigma_W(\cdot)$. To present our results we first provide the necessary formal definitions. Similar to the degree sequence, the *weight sequence* of a vertex-weighted graph is simply the non-increasing sequence of vertex weights.

Definition 2 (Greedy vertex-weighted trees)**.** With given degree sequence and weight sequence, the greedy vertex-weighted tree is defined through the following algorithm:

(i) Label the vertex with the largest degree as v (the root) and assign to v the largest weight;
(ii) Label the neighbors of v as $v_1, v_2, \ldots$, assign the largest available degrees and weights to them such that $\deg(v_1) \geq \deg(v_2) \geq \cdots$ and $w(v_1) \geq w(v_2) \geq \cdots$;
(iii) Label the neighbors of v_1 (except v) as $v_{11}, v_{12}, \ldots$ such that they take all the largest available degrees and weights, with $\deg(v_{11}) \geq \deg(v_{12}) \geq \cdots$ and $w(v_{11}) \geq w(v_{12}) \geq \cdots$, then do the same for $v_2, v_3, \ldots$;
(iv) Repeat (iii) for all the newly labeled vertices, always start with the neighbors of the labeled vertex with largest degree (weight) whose neighbors are not labeled yet.

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