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Riemann–Hilbert approach for an initial-boundary value problem of the two-component modified Korteweg-de Vries equation on the half-line

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ABSTRACT

In this work, we investigate the two-component modified Korteweg-de Vries (mKdV) equation, which is a complete integrable system, and accepts a generalization of 4×4 matrix Ablowitz–Kaup–Newell-Segur (AKNS)-type Lax pair. By using of the unified transform approach, the initial-boundary value (IBV) problem of the two-component mKdV equation associated with a 4×4 matrix Lax pair on the half-line will be analyzed. Supposing that the solution $\{u_1(x,t), u_2(x,t)\}$ of the two-component mKdV equation exists, we will show that it can be expressed in terms of the unique solution of a 4×4 matrix Riemann–Hilbert problem formulated in the complex λ-plane. Moreover, we will prove that some spectral functions $s(\lambda)$ and $S(\lambda)$ are not independent of each other but meet the global relationship.

1. Introduction

It is well known that the modified Korteweg-de Vries (mKdV) equation is one of the most important integrable systems in mathematics and physics which reads

$$u_t + u_{xxx} + 6\epsilon u^2 u_x = 0, \quad \epsilon = \pm 1, \tag{1.1}$$

and which is a natural extension of the KdV equation, whose can be describe acoustic wave in a certain anharmonic lattice and the Alfven wave in a cold collision-free plasma. The mKdV equation have been studied extensively because of their physical significance [1–5].

Currently, much attention is paid to the multi-component equations [3,6–8] for their important physical applications in many fields, such as fluid mechanics, nonlinear optics to Bose–Einstein condensates, and field theories [6–8]. In addition, multi-component equations possess abundant solution structures and appealing soliton collision phenomena [9–11]. For instance, when the polarization effect is included in the problem of pulse propagation along the optical fiber, the scalar nonlinear Schrödinger (NLS) [12] equation which governs the propagation of light pulses with fixed polarization should be replaced by a two-component generalization of the NLS equation.

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Like the NLS equation, generally speaking, there exist two categories of the mKdV equation. One example is a vector version of the mKdV equation proposed by Yajima and Oikawa [2], the vector mKdV (vmKdV) equation has been studied extensively on the integrability associated with explicit form of the Lax pair [13], the initial value problem [9], infinitely many conservation laws [14], soliton solutions given by the Darboux transformation (DT) [14] and by the bilinear approach [7] and by the inverse scattering transform (IST) [9], etc. Besides, the algebro-geometric solutions of the vmKdV equation associated with a 3×3 matrix spectral problem on the basis of the theory of algebraic curves [15].

An other example is a matrix version of the mKdV equation studied by Athorne and Fordy [3]. In 1997, Iwao and Hirota [7] discussed a simple coupled version of the mKdV equation

$$u_{j,t}+6\left(\sum_{k,l=1}^{N}c_{kl}u_ku_l\right)u_{j,x}+u_{j,xxx}=0,\quad j=1,2,\ldots,N, \tag{1.2}$$

it is not difficulty to find that system (1.2) can be transformed into the following normalized multi-component mKdV equations [8]

$$u_{j,t}+6\left(\sum_{k=1}^{N}\epsilon_k u_k^2\right)u_{j,x}+u_{j,xxx}=0,\quad \epsilon_k=\pm 1,\quad j=1,2,\ldots,N. \tag{1.3}$$

Recently, the initial-boundary value (IBV) problems for the mKdV equation on the half-line are studied via the unified transform approach [5], the unified transform approach can be used to study the IBV problems for both linear and nonlinear integrable evolution PDEs with 2×2 Lax pairs [16–22]. Just like the IST on the line, the unified transform method gets an expression for the solution of an IBV problem in terms of the solution of a Riemann–Hilbert (RH) problem. In 2012, Lenells extended the unified transform method to study the IBV problems for integrable nonlinear equations with 3×3 Lax pairs [23]. After that, many researchers have devoted their attention to studying IBV problems for integrable evolution equations with higher-order Lax pairs, the IBV problem for the many integrable system with 3×3 or 4×4 Lax pairs are studied, such as, the Degasperis–Procesi equation [24,25], the Ostrovsky–Vakhnenko equation [26], the Sasa–Satsuma equation [27], the three wave equation [28], the spin-1 Gross–Pitaevskii equations [29] and others [30–33]. These authors have also done some work on integrable equations with 2×2 or higher-order Lax pairs [21,34–36]. In particular, an effective way analyzing the asymptotic behaviour of the solution is based on this RH problem and by using the nonlinear version of the steepest descent method introduced by Deift and Zhou [37].

Most recently, Geng et al. [13,38] discussed the IBV problem of the coupled mKdV (vector version) equation on the half-line. To the best of our knowledge, the IBV problem of the two-component mKdV equation (simple matrix version) has not been studied. In this work, by using the unified transform method, we investigate the self-focusing type case of two-component mKdV equation which reads

$$\begin{cases}u_{1,t}+6(u_1^2+u_2^2)u_{1,x}+u_{1,xxx}=0,\\ u_{2,t}+6(u_1^2+u_2^2)u_{2,x}+u_{2,xxx}=0.\end{cases} \tag{1.4}$$

That is to say, we consider the following initial-boundary value problem of the two-component mKdV equation. The IBV problems of system (1.4) on the interval will be presented in another paper.

Throughout this paper, we consider the half-line domain Ω and the IBV problems for system (1.4) as follows

$$\begin{aligned}&\text{Half-line domain (see Figure 1)}:\ \Omega=\{0<x<\infty, 0<t<T\};\\ &\text{Initial values}:\ u_0(x)=u_1(x,t=0),\ v_0(x)=u_2(x,t=0);\\ &\text{Dirichlet boundary values}:\ p_0(t)=u_1(x=0,t),\ q_0(t)=u_2(x=0,t);\\ &\text{Neumann boundary values}:\ p_1(t)=u_{1,x}(x=0,t),\ q_1(t)=u_{2,x}(x=0,t).\end{aligned} \tag{1.5}$$

Here $u_0(x)$ and $v_0(x)$ belong to the Schwartz space.

The outline of the present paper is as follows. In Section 2, two sets of eigenfunctions $\{\mu_j\}_1^3$ and $\{M_n\}_1^4$ of the Lax pair for spectral analysis are given, and further determine some spectral functions which meet a so-called global relationship. In Section 3, we prove that $\{u_1(x, t), u_2(x, t)\}$ can be described in terms of the unique solution of a 4×4 matrix Riemann–Hilbert problem formulated in the complex λ-plane. The last section is devoted to giving some conclusions.

2. The spectral analysis

The two-component mKdV Eq. (1.4) admits the 4×4 Lax pair

$$\begin{cases}\psi_x=F\psi=(-i\lambda\Lambda+F_0)\psi,\\ \psi_t=G\psi=(-4i\lambda^3\Lambda+4\lambda^2F_0+2i\lambda G_1+G_0)\psi,\end{cases} \tag{2.1}$$