Applied Mathematics and Computation 333 (2018) 480–489

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Edge-partition and star chromatic index

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ARTICLE INFO

Keywords:
Star chromatic index
Strong chromatic index
Planar graph
Outerplanar graph
Edge-partition

ABSTRACT

The star chromatic index $\chi'_{st}(G)$ of a graph G is the smallest integer k for which G has a proper edge k-coloring without bichromatic paths or cycles of length four. The strong chromatic index $\chi'_s(G)$ of G is the smallest integer k for which G has a proper edge k-coloring such that any two edges at distance at most two get distinct colors.

In this paper, we prove that if a graph G can be edge-partitioned into two graphs F and H, then $\chi'_{st}(G) \le \chi'_{st}(F) + \chi'_s(H|_G)$, where $\chi'_s(H|_G)$ denotes the strong chromatic index of H restricted on G. Using this result, we give some upper bounds of the star chromatic index for planar graphs. Precisely, we show that (1) if G is a planar graph with maximum degree Δ, then $\chi'_{st}(G) \le 2.75\Delta + 18$; (2) if G is a planar graph without 4-cycles, then $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 18$; (3) if G is a planar graph of girth at least 8, then $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 3$; (4) if G is a K_4-minor free graph, then $\chi'_{st}(G) \le 2.25\Delta + 6$; and (5) if G is an outerplanar graph, then $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 5$, which improves a result in Bezegová et al. [2], which says that $\chi'_{st}(G) \le \lfloor 1.5\Delta \rfloor + 12$ for an outerplanar graph G.

1. Introduction

Only simple graphs are considered in this paper. For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and minimum degree of G, respectively. If no ambiguity arises in the context, $\Delta(G)$ is simply written as Δ. A vertex v is called a *k-vertex* (*k^+-vertex*, *k^--vertex*, respectively) if the degree of v is k (at least k, at most k, respectively). Let $\sigma(G) = |V(G)| + |E(G)|$. A vertex of degree 1 is called a *leaf*. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G.

A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges, and G is called *H-minor free* if G does not have H as a minor. A graph G is *planar* if it can be drawn on the Euclidean plane so that any two edges intersect only at their ends. A planar graph G is *outerplanar* if it admits a plane embedding such that all its vertices lie on the boundary of the same face. It is shown in [5] that a graph G is an outerplanar graph if and only if G is K_4-minor free and $K_{2,3}$-minor free. Thus, the class of outerplanar graphs is a special class of K_4-minor free graphs.

A *proper edge k-coloring* of a graph G is a mapping $\phi : E(G) \to \{1, 2, \dots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. The *chromatic index* $\chi'(G)$ of G is the smallest integer k such that G has a proper edge k-coloring.

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[1] Research supported by NSFC (No. 11671053).
[2] Research supported by NSFC (No. 11771402).

https://doi.org/10.1016/j.amc.2018.03.079
0096-3003/

A proper edge k-coloring ϕ of G is called a *strong-edge k-coloring* if any two edges of distance at most two get distinct colors. That is, each color class is an induced matching in the graph G. The *strong chromatic index*, denoted $\chi_s'(G)$, of G is the smallest integer k such that G has a strong-edge k-coloring.

A proper edge k-coloring ϕ of G is called a *star-edge k-coloring* if there do not exist bichromatic paths or cycles of length four. That is, at least three colors are needed for a path or a cycle of length four. The *star chromatic index*, denoted $\chi_{st}'(G)$, of G is the smallest integer k such that G has a star-edge k-coloring.

From the definitions, it is straightforward to see that $\chi_s'(G) \geq \chi_{st}'(G) \geq \chi'(G) \geq \Delta$.

The strong-edge coloring of graphs was introduced by Fouquet and Jolivet [9]. Erdős and Nešetřil put forward the following challenging conjecture:

Conjecture 1. *Let G be a simple graph. Then*

$$\chi_s'(G) \leq \begin{cases} 1.25\Delta^2, & \text{if } \Delta \text{ is even;} \\ 1.25\Delta^2 - 0.5\Delta + 0.25, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Using probabilistic method, Molloy and Reed [13] showed that $\chi_s'(G) \leq 1.998\Delta^2$ for a graph G with sufficiently large Δ. This result was furthermore improved in [3] to that $\chi_s'(G) \leq 1.93\Delta^2$ for any graph G. For the class of planar graphs, by using the Four-Color Theorem [1] and Vizing Theorem [20], Faudree et al. [7] gave an elegant and short proof to the result that $\chi_s'(G) \leq 4\Delta + 4$. Moreover, they constructed a class of planar graphs G with $\Delta \geq 2$ so that $\chi_s'(G) = 4\Delta - 4$.

The star-edge coloring of a graph G was initiated by Liu and Deng [12]. It was declared in [12] that every simple graph G with $\Delta \geq 7$ satisfies $\chi_{st}'(G) \leq \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$. Recently, Dvořák, Mohar and Šámal [6] considered the star chromatic indices of complete graphs, subcubic graphs, etc. Among other things, they showed the following results and raised a conjecture:

Theorem 1. *([6]) Let G be a simple graph with maximum degree Δ and $K_{\Delta+1}$ be a complete graph of order $\Delta + 1$. Then*

$$\chi_{st}'(K_{\Delta+1}) \leq \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log(\Delta+1)}}}{(\log(\Delta+1))^{\frac{1}{4}}}(\Delta+1),$$

$$\chi_{st}'(G) \leq O\left(\frac{\log \Delta}{\log\log \Delta}\right)^2 \cdot \chi_{st}'(K_{\Delta+1}),$$

and therefore $\chi_{st}'(G) \leq \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}$.

Theorem 2. *([6]) Let G be a subcubic graph. Then*
(1) $\chi_{st}'(G) \leq 7$.
(2) If G is simple and cubic, then $\chi_{st}'(G) \geq 4$, and the equality holds if and only if G covers the graph of 3-cube.

Conjecture 2. *([6]) Every subcubic graph G has $\chi_{st}'(G) \leq 6$.*

Bezegová et al. [2] investigated the star-edge coloring of trees and outerplanar graphs by showing the following results:

Theorem 3. *([2]) If T is a tree, then $\chi_{st}'(G) \leq \lfloor 1.5\Delta \rfloor$. Moreover, the upper bound is tight.*

Theorem 4. *([2]) Let G be an outerplanar graph. Then*
(1) $\chi_{st}'(G) \leq \lfloor 1.5\Delta \rfloor + 12$.
(2) If $\Delta \leq 3$, then $\chi_{st}'(G) \leq 5$.

In [2], the authors pointed out that the constant 12 in Theorem 4(1) can be decreased to 9 by using more involved analysis. Moreover, they put forward the following conjecture:

Conjecture 3. *([2]) Every outerplanar graph G with $\Delta \geq 3$ has $\chi_{st}'(G) \leq \lfloor 1.5\Delta \rfloor + 1$.*

In this paper, we investigate the star-edge coloring of planar graphs by using an edge-partition technique and a useful relation between star chromatic index and strong chromatic index. Precisely, for a planar graph G with maximum degree Δ and girth g, we prove the following results:

- $\chi_{st}'(G) \leq 2.75\Delta + 18$.
- $\chi_{st}'(G) \leq 2.25\Delta + 6$ if G is K_4-minor free.
- $\chi_{st}'(G) \leq \lfloor 1.5\Delta \rfloor + 18$ if G has no 4-cycles.
- $\chi_{st}'(G) \leq \lfloor 1.5\Delta \rfloor + 13$ if $g \geq 5$.
- $\chi_{st}'(G) \leq \lfloor 1.5\Delta \rfloor + 3$ if $g \geq 8$.
- $\chi_{st}'(G) \leq \lfloor 1.5\Delta \rfloor + 5$ if G is outerplanar.

Note that the last result above is an improvement to the result (1) in Theorem 4.