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Two-grid stabilized methods for the stationary incompressible Navier–Stokes equations with nonlinear slip boundary conditions☆

Hailong Qiu

School of Mathematics and Physics, Yancheng Institute of Technology, Yancheng 224051, China

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ABSTRACT

In this paper, we consider a two-grid quadratic equal-order stabilized method for the stationary incompressible Navier–Stokes equations with nonlinear slip boundary conditions. Our two-grid stabilized method consists of computing one nonlinear problem on a coarse mesh and then solving a linearization correction problem on a fine mesh. Moreover, the stability and convergence of the present method are derived. Finally, numerical experiments are performed to confirm our theoretical results.

1. Introduction

Incompressible viscous flow with nonlinear slip boundary conditions has been successfully applied to some flow phenomena in environmental and medical problems such as oil flow over or beneath sand layer and blood flow in the thoracic aorta [5].

The governing equations are the incompressible Navier–Stokes ones:

$$\begin{cases} -\mu\Delta u + (u\cdot\nabla)u + \nabla p = f, & \text{in } \Omega, \\ divu = 0, & \text{in } \Omega, \end{cases} \tag{1}$$

with the following friction boundary conditions:

$$\begin{cases} u = 0, & \text{on } \Gamma, \\ u_n = 0, \quad -\sigma_\tau(u) \in g\partial|u_\tau|, & \text{on } S, \end{cases} \tag{2}$$

where Ω is a bounded convex region in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$ partitioned into nonoverlapping parts, i.e., $\Gamma\cap S = \emptyset$, $\overline{\Gamma\cup S} = \partial\Omega$. $u = (u_1(x), u_2(x))$ (or $u = (u_1(x), u_2(x), u_3))$ is the velocity vector, $p = p(x)$ the pressure. $f = (f_1(x), f_2(x))$ (or $f = (f_1(x), f_2(x), f_3))$ stands for some body force vector, g is a scalar function and $\mu > 0$ is the viscous coefficient. We assume that $u_n = u\cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, where n stands for the unit vector of the external normal to S. $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of p, is the tangential component of the stress vector σ which is defined by $\sigma_i = \sigma_i(u, p) = (\mu e_{ij}(u) - p\delta_{ij})n_j$, where $e_{ij}(u) = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}$, $i, j = 1, 2$.

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E-mail addresses: qhllf@126.com, hail_qiu@126.com

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We assume that the set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a. From the definition of the subdifferential in the following section, we can know that the variational formulation of (1) and (2) is the variational inequality problem of the second kind.

For the nonlinear slip boundary conditions (2), Fujita in [6] investigated existence and uniqueness of weak solution to the Stokes problem. Subsequently, Saito [23] studied regularity of the weak solution by Yoshidas regularized method and difference quotients. Meanwhile, the existence and uniqueness of the weak solution to the Navier–Stokes problem under nonlinear boundary conditions were obtained in [15,19]. For numerical methods for the Navier–Stokes equations, Li and An in [16] studied pressure stabilized finite element method for this type of boundary conditions. Kashiwabara in [10] studied discrete variational inequality problem for the Stokes equations with leak boundary conditions of friction type. Other theoretical and numerical results for the steady and unsteady Stokes/Navier–Stokes problems with such boundary conditions can be found in [17,18,20–22].

Although the available computing power is developing very fast nowdays, solving the Navier–Stokes equations numerically is still remain some important but challenging problems. For instance, applying finite element methods to compute the Navier–Stokes equations may bring out several shortcomings: the oscillations due to the dominating convection, the violation of the discrete inf-sup condition and so on, which can all lead numerical computation to a large computational scale. Hence, to increase the efficiency of numerical algorithms, to the authors' knowledge, two-grid method is a simple but efficient method. The main idea of two-grid discretization strategy is to capture the low modes, or global solution envelope by solving an initial approximation problem on a coarse mesh. The fine structures are captured by computing one linearized system on a fine mesh. Some details of two-grid method can be found in the works of Layton et al. [11,12], Ervin et al. [3], He et al. [7,8].

In the recent years, He and Li [9] studied a two-grid method for the steady Navier–Stokes problem applying the mini-element, which satisfies the discrete inf-sup condition. However, due to computational convenience and efficiency in practice, some finite element pairs which do not satisfy the discrete inf-sup condition are also popular. Thus, much attention has been paid to the study of the stabilized method for the Stokes and Navier–Stokes problems. Recently, a new family of stabilized method based on polynomial pressure projection was proposed [1,2], which has some prominent features: without higher-order derivatives and edge-based data structures, and stabilization being completely local at the element grid. Moreover, based on the work of Bochev, Li et al. [1,2,13] have studied on stabilization of the lowest equal-order finite element pair $P_1 - P_1$ using the projection of the pressure onto the piecewise constant space, of the quadratic equal-order finite element pair $P_2 - P_2$ using the projection of the pressure-gradient onto the piecewise constant space [24]. these methods obtain that the discrete inf-sup condition holds by adding to a bilinear form. Therefore, it is gaining more and more popularity in computational fluid dynamics.

The purpose of this paper is to consider two-grid stabilized method for the incompressible Navier–Stokes equations with friction boundary conditions. The basic idea of our method is to solve an initial solution on a coarse mesh, then to compute a linearized correction system on a fine mesh. More specifically, our two-grid stabilized method includes three algorithms: Stokes correction algorithm, Oseen correction algorithm and Newton correction algorithm. Denote (u^h, p^h) the Stokes/Oseen/Newton correction algorithm solutions to the problem (1) and (2). Then for Stokes correction algorithm and Oseen correction algorithm, there hold

$$\|u - u^h\|_V + \|p - p^h\| \le c(h^2 + H^3).$$

For Newton correction algorithm, there hold

$$\|u - u^h\|_V + \|p - p^h\| \le c(h^2 + H^4).$$

This paper is organized as follows. In next Section, we will introduce some notations, and give the variational formulation of the problem (1) and (2). In Section 3, we will recall stabilized finite element method. Three two-grid stabilized correction algorithms are shown, the stability and convergence of the presented algorithms are established in Section 4. The numerical results is given in last Section.

Throughout this paper, the letter c always denotes some positive constant independent of the mesh parameter h, H and may be different even in the same formulation.

2. Preliminaries

As in [23], for the mathematical setting of problem (1) and (2), we introduce the definition of the subdifferential property [4]. Let $\psi : \mathbb{R}^d \to \overline{R} = (-\infty, +\infty]$ be a given function possessing the properties of convexity and weak semi-continuity from below (ψ is not identical with $+\infty$). We say that the set $\partial\psi(a)$ is a subdifferential of the function ψ at the point a if and only if

$$\partial\psi(a) = \{b \in \mathbb{R}^d : \psi(h) - \psi(a) \ge b \cdot (h - a), \quad \forall h \in \mathbb{R}^d\}.$$

Introduce the following Hilbert space:

$$V = \{u \in H^1(\Omega)^d,\ u|_\Gamma = 0,\ u \cdot n|_S = 0\}, \quad V_0 = H_0^1(\Omega)^d,$$

$$V_\sigma = \{u \in V, \quad \nabla \cdot u = 0\}, \qquad M = L_0^2(\Omega) = \left\{q \in L^2(\Omega), \int_\Omega q dx = 0\right\}.$$