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Neighbor sum distinguishing total chromatic number of planar graphs

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ABSTRACT

Let $G = (V(G), E(G))$ be a graph and ϕ be a proper k-total coloring of G. Set $f_\phi(v) = \sum_{uv\in E(G)} \phi(uv) + \phi(v)$, for each $v \in V(G)$. If $f_\phi(u) \neq f_\phi(v)$ for each edge $uv \in E(G)$, the coloring ϕ is called a k-neighbor sum distinguishing total coloring of G. The smallest integer k in such a coloring of G is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_\Sigma(G)$. In this paper, by using the famous Combinatorial Nullstellensatz, we determine $\chi''_\Sigma(G)$ for any planar graph G with $\Delta(G) \geq 13$.

1. Introduction

The terminology and notation used but undefined in this paper can be found in [2]. In this paper, we only consider simple, finite and undirected graphs. Let $G = (V(G), E(G))$ be a graph with maximum degree $\Delta(G)$. Let $d_G(v)$ or simply $d(v)$ denote the degree of a vertex v in G. A vertex v is a $t-vertex$ ($t^+-vertex$, $t^- - vertex$) if $d(v) = t$ ($d(v) \geq t$, $d(v) \leq t$). Let $d_t(v)$ ($d_{t^-}(v)$, $d_{t^+}(v)$) denote the number of t-vertices (t^--vertices, t^+-vertices) adjacent to v. Let $N(v)$ denote the neighbor set of the vertex v of G. Let $G = (V(G), E(G), F(G))$ be a plane graph. An l-face $f = v_1v_2\ldots v_l \in F(G)$ is a $(b_1, b_2, \ldots, b_l)$*-face*, if v_i is a b_i-vertex, for $i = 1, 2, \ldots, l$.

Given a graph G and a proper k-total coloring $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$. The coloring ϕ is a *k-neighbor sum (set) distinguishing total coloring* if $f_\phi(u) \neq f_\phi(v)(S_\phi(u) \neq S_\phi(v))$ for each edge $uv \in E(G)$, where $f_\phi(v)(S_\phi(v))$ is the sum (set) of colors on the edges incident with v and the color on the vertex v. The smallest number k in such a coloring of graph G is the *neighbor sum (set) distinguishing total chromatic number*, denoted by $\chi''_\Sigma(G)(\chi''_a(G))$. Clearly, for any graph G, $\chi''_a(G) \leq \chi''_\Sigma(G)$. All colorings considered in this paper are proper colorings. Zhang et al. [25] proposed the following conjecture.

Conjecture 1.1 [25]**.** *For any graph G, $\chi''_a(G) \leq \Delta(G) + 3$.*

Huang and Wang [8] showed that Conjecture 1.1 holds for planar graphs with $\Delta(G) \geq 11$, which was extended to $\Delta(G) \geq 10$ by Cheng et al. [4]. Wang and Huang [21] showed that if G is a planar graph with $\Delta(G) \geq 13$, then $\chi''_a(G) \leq \Delta(G) + 2$, meanwhile, they proved that if G has no adjacent $\Delta(G)$-vertices, then $\chi''_a(G) = \Delta(G) + 1$.

For k-neighbor sum distinguishing total coloring (or simply k-tnsd-coloring), Pilśniak and Woźniak [12] gave the following conjecture.

Conjecture 1.2 [12]**.** *For any graph G, $\chi''_\Sigma(G) \leq \Delta(G) + 3$.*

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Since $\chi''_a(G) \le \chi''_\Sigma(G)$ for any graph G, Conjecture 1.2 implies Conjecture 1.1. Conjecture 1.2 was confirmed for complete graphs, cycles, bipartite graphs and subcubic graphs in [12]. Dong and Wang [5] showed that Conjecture 1.2 holds for some sparse graphs. Yao et al. [23,24] considered tnsd-coloring of degenerate graphs. [6,14] considered the list version of tnsd-coloring of graphs. [7,10,11,17–20] considered tnsd-coloring of planar graphs with cycle restrictions. Song and Xu [16] determined $\chi''_\Sigma(G)$ for any K_4-minor free graph G with $\Delta(G) \ge 5$. Li et al. [9] showed that Conjecture 1.2 holds for planar graphs with maximum degree at least 13, and subsequently the result was improved by Qu et al. [13] and by Yang et al. [22]. Cheng et al. [3] showed that if G is a planar graph with $\Delta(G) \ge 14$, then $\chi''_\Sigma(G) \le \Delta(G)+2$. Recently, Song et al. [15] improved this result and get that:

Theorem 1.1 [15]**.** *Let G be a planar graph with maximum degree $\Delta(G)$. Then $\chi''_\Sigma(G) \le max\{\Delta(G)+2, 14\}$.*

In this paper, we get the following results.

Theorem 1.2. *Let G be a planar graph without adjacent maximum degree vertices. Then $\chi''_\Sigma(G) \le max\{\Delta(G)+1, 14\}$.*

Since $\chi''_\Sigma(G) \ge \Delta(G)+1$ for any graph G, and if G has adjacent $\Delta(G)$-vertices, then $\chi''_\Sigma(G) \ge \Delta(G)+2$. Thus we get the following corollary.

Corollary 1.1. *Let G be a planar graph and $\Delta(G) \ge 13$. If G has no adjacent $\Delta(G)$-vertices, then $\chi''_\Sigma(G) = \Delta(G)+1$, otherwise $\chi''_\Sigma(G) = \Delta(G)+2$.*

Clearly, this result implies the result in [21].

2. Preliminaries

For any graph G, set $n_i(G) = |\{v \in V(G) \mid d_G(v) = i\}|$ for each positive integer. A graph G' is *smaller* than G if one of the following holds:

(1) $|E(G')| < |E(G)|$;
(2) $|E(G')| = |E(G)|$ and $(n_t(G'), n_{t-1}(G'), \dots, n_1(G'))$ precedes $(n_t(G), n_{t-1}(G), \dots, n_1(G))$ with respect to the standard lexicographic order, where $t = \max\{\Delta(G), \Delta(G')\}$.

A graph is *minimum* for a property when no smaller graph satisfies it.

Let $P = P(x_1, x_2, \dots, x_n)$ be a polynomial in n $(n \ge 1)$ variables. Set $c_P(x_1^{k_1} x_2^{k_2} \dots x_n^{k_n})$ be the coefficient of the monomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in P, where k_i $(1 \le i \le n)$ is a non-negative integer. We need the famous Combinatorial Nullstellensatz to get our main result.

Lemma 2.1 [1]**.** (*Combinatorial Nullstellensatz*). *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, x_2, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \dots, x_n]$. Suppose the degree of P equals $\sum_{i=1}^n k_i$, where each k_i is a nonnegative integer, and suppose $c_P(x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}) \ne 0$. If $S_1, \dots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$, there are $s_1 \in S_1, \dots, s_n \in S_n$ so that $P(s_1, \dots, s_n) \ne 0$.*

3. The proof of Theorem 1.2

We will prove Theorem 1.2 by contradiction. Let G be a minimum counterexample of Theorem 1.2. Let $k = \max\{\Delta(G)+1, 14\}$. By the choice of G, any planar graph G' without adjacent $\Delta(G')$-vertices which is smaller than G has a k-tnsd-coloring ϕ'. In the following, we will choose some planar graph G' smaller than G and extend the coloring ϕ' of G' to the desired coloring ϕ of G to obtain a contradiction. Without special remark, for any $x \in (V(G) \cup E(G)) \cap (V(G') \cup E(G'))$, let $\phi(x) = \phi'(x)$. For a vertex $v \in V(G)$ with $d(v) = l$, we usually set $N(v) = \{v_1, v_2, \dots, v_l\}$.

Noting that when we color a 4^--vertex, it has at most 12 forbidden colors and $k \ge 14$, so we can color it easily. Thus in the following proof, we will omit the colors of all 4^--vertices. In all figures of this paper, the degrees of black vertices are shown and the degrees of open vertices are at least shown.

Claim 1. *For each vertex $v \in V(G)$, if $d_1(v) \ge 1$, then $d_2(v) = 0$.*

Proof. Suppose to the contrary that G has a vertex v which is adjacent to a 1-vertex v_1 and a 2-vertex v_2. Let $N(v_2) = \{u, v\}$ and G' be the graph by splitting v_2 into v_2 and v'_2 (see Fig. 1). Thus G' has a k-tnsd-coloring ϕ'. For each $x \in V(G) \cup E(G) \setminus \{v_2u, vv_1, vv_2, v_1, v_2\}$, set $\phi(x) = \phi'(x)$. And set $\phi(v_2u) = \phi'(v'_2u)$, $\phi(vv_2) \in \{\phi'(vv_1), \phi'(vv_2)\} \setminus \{\phi(v_2u)\}$, $\phi(vv_1) \in \{\phi'(vv_1), \phi'(vv_2)\} \setminus \{\phi(vv_2)\}$. v_1, v_2 are 2^--vertices. We can get a k-tnsd-coloring ϕ of G, a contradiction. □

Claim 2. *Each t^--vertex is not adjacent to any $(10-t)^-$-vertex in G, where $t = 5, 6$.*

Proof. On the contrary, we assume that a t-vertex u is adjacent to a $(10-t)$-vertex v for $t = 5, 6$ (other cases can be proved easier). Let $N(u) = \{v, u_1, \dots, u_{t-1}\}$, $N(v) = \{u, v_1, \dots, v_{9-t}\}$. Let $G' = G - uv$, then G' has a k-tnsd-coloring ϕ'. First, delete the colors of u and v. Next we will color uv and recolor u, v. Let S_1, S_2, S_3 be the sets of available colors for u, uv, v. Since