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A two-dimensional Chebyshev wavelets approach for solving the Fokker-Planck equations of time and space fractional derivatives type with variable coefficients

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ABSTRACT

In the current study, we consider the numerical solutions of the Fokker-Planck equations of time and space fractional derivative type with variable coefficients. The proposed method is based on the two-dimensional Chebyshev wavelet basis together with their corresponding operational matrices of fractional-order integration. The convergence analysis of the proposed method is rigorously established. Numerical tests are carried out to confirm the effectiveness and feasibility of the proposed scheme.

1. Introduction

The Fokker-Planck equation was first introduced by Fokker and Planck to describe the Brownian motion of particles [1], that is, it expresses the change of probability of a random function in space and time, hence it is used to explain solute transport. These equations have shown their applications in diverse scientific areas, including biology [2], physics [3,4] and finance [5]. The Fokker-Planck (FP) equations for stochastic dynamical systems [6–8] describe the time evolution for the probability density of solution paths. The space fractional Fokker-Planck equation origins from the continuous time random walk with the Levy jumping length distribution, while the time fractional Fokker-Planck (FP) equation arises from the continuous time random walk of Mittag–Leffler waiting time distribution. Particularly, for describing the interaction between subdiffusion and time-modulation, a physically realistic time-dependent time fractional FP equation has been derived from the CTRW with time-modulated Boltzmann jumping probabilities [9–13].

Many attempts have been made to develop the approximate methods to solve fractional partial differential equations, such as RBF meshless method [14], FDM [15], HAM [16], flatlet oblique multiwavelets method [17] and operational matrix method [18]. For the solutions of linear and nonlinear Fokker-Planck equations, these methods include ADM [19], HPM [20, 21], VIM [22,23], RBFs method [24], B-spline scaling functions method [25]. For numerical algorithms and solutions details can be found in [26,27]. In Ref. [28], the authors applied spectral collocation method to obtain the numerical solutions of time fractional Fokker-Planck equations. In Ref. [29], the authors studied the numerical solutions of time-space fractional Fokker-Planck equation with variable force field and diffusion. An efficient computational technique for solving the Fokker-Planck equation with space and time fractional derivative is proposed in Ref. [30]. In this paper, we applied the

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two-dimensional Chebyshev wavelet for solving two kinds of fractional Fokker-Planck equations with variable coefficients. Numerical results shown that our method is effective and robust.

The paper is organized as follows: Section 2 introduces the basic definitions of fractional derivative and integration. Section 3 gives the properties of the Chebyshev wavelet and their operational matrix of fractional integration. Section 4 and Section 5 apply the proposed method for solving the Fokker-Planck equations of time and space fractional derivatives type with variable coefficient, respectively. Several numerical examples are provided to test the proposed scheme in Section 6. A conclusion is drawn in Section 7.

2. Fractional derivative and integration

Definition 2.1. The most frequently encountered definition of fractional order is the Riemann-Liouville integral in which the fractional integral operator I^α $(\alpha > 0)$ of a function $f(t)$, is defined as [31]

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \left(\alpha > 0 \text{ and } \alpha \in \Re^+\right), \tag{1}$$

where $\Gamma(\,\bullet\,)$ is the well-known gamma function, and some properties of the operator I^α are as follows

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \quad (\alpha > 0, \beta > 0), \tag{2}$$

$$I^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\alpha+\gamma}, \quad (\gamma > -1). \tag{3}$$

Definition 2.2. The fractional derivative introduced by Caputo, in the late sixties, is called Caputo fractional derivative. The Caputo fractional derivative ${}_cD_t^\alpha$ of a function $f(t)$ is defined as [31]

$${}_cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \quad (n-1 < \alpha \le n,\ n \in N). \tag{4}$$

The following are two basic properties of the Caputo fractional derivative

$${}_cD_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad (0 < \alpha < \beta+1, \quad \beta > -1), \tag{5}$$

$$I^\alpha {}_cD_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}\left(0^+\right) \frac{t^k}{k!}, \quad (\,n-1 < \alpha \le n \text{ and } n \in N). \tag{6}$$

3. Chebyshev wavelets and their properties

The Chebyshev wavelets $\psi_{nm}(t) = \psi(k, n, m, t)$ have four arguments, $n = 1, 2, \ldots, 2^{k-1}$, $k \in N^*$. t is the normalized time. They are defined on the interval $[0, 1)$ as [32]

$$\psi_{nm}(t) = \begin{cases} 2^{k/2} \tilde{T}_m\left(2^k t - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & o.w. \end{cases} \tag{7}$$

with

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}} T_m(t), & m = 1, 2, \ldots, M-1. \end{cases} \tag{8}$$

Here $T_m(t)$ are the Chebyshev polynomials with the weight function $\omega(t) = 1/\sqrt{1-t^2}$ and satisfy the following recursive formula

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1, 2, \ldots$$

A function $f(t)$ defined over $L^2_{\omega_n}[0, 1)$ may be expanded as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \tag{9}$$

where

$$c_{nm} = \langle f(t), \psi_{nm}(t) \rangle_{\omega_n} = \int_0^1 \omega_n(t) \psi_{nm}(t) dt, \tag{10}$$