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A stable explicitly solvable numerical method for the Riesz fractional advection–dispersion equations

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ABSTRACT

In this paper, we present a finite difference scheme for solving the Riesz fractional advection-dispersion equations (RFADEs). The scheme is obtained by using asymmetric discretization technique and modify the shifted Grünwald approximation to fractional derivative. By calculating the unknowns in differential nodal-point sequences at the odd and even time-levels, the discrete solution of the scheme can be obtained explicitly. The computational cost for the scheme at each time step can be $O(K\log K)$ by using the fast matrix-vector multiplication with the help of Toeplitz structure, where K is the number of unknowns. We prove that the scheme is solvable and unconditionally stable. We derive the error estimates in discrete l^2-norm, which is optimal in some cases. Numerical examples are presented to verify our theoretical results.

1. Introduction

Fractional differential equations (FDEs) are getting more and more attentions in various fields, such as in physics [4,23,36], in hydrology [1,3,33], in biology [20] and chemistry [16], as well as in finance [29,32]. FDEs can be used to describe phenomena of anomalous diffusion which can not be modelled accurately by second-order diffusion equations. For instance, in contaminant transport in groundwater flow the solutes moving through aquifer do not generally follow a second-order diffusion equation because of a large deviations from the stochastic process of Brownian motion. A governing equation with fractional-order anomalous diffusion provides a more adequate and accurate description of the movement of the solutes [6]. Hence, efficiently solving the FDEs becomes a urgent topic.

Since the fractional-order differential and integral operators are non-local [26], it is difficult to seek analytical solutions by using classical methods, such as the Fourier transform method, and the Laplace transform method. Therefore, people shift their focus to the study of numerical methods for FDEs.

In the last decades, a variety of numerical methods have been developed for solving FDEs, including finite element methods [6,9], finite different methods [5,21,24], spectral methods [17,40], finite volume methods [12–14] and so on. Discretization methods and corresponding convergence analysis were established. For example, in [21,22] Meerschaert and Tadjeran built a native discretization of fractional advection-dispersion equations (FADEs) by using shifted Grünwald discretization. The implicit scheme was proved to be stable, convergent and with first-order accuracy in space meshsize. Moreover, several explicit finite difference schemes, such as the fractional central and Lax-Wendroff schemes, were proposed and analyzed in [37]. These methods are conditionally stable.

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The RFADEs is one kind of FDEs. It is obtained from the standard advection-dispersion equation by replacing the first-order and the second-order space derivatives with the Riesz fractional derivatives of order $0<\beta\le 1$ and of order $1<\alpha\le 2$, respectively [30,41]. Shen and Liu [34] presented some numerical solutions for the RFADE. Yang et al.[39] proposed some numerical methods for fractional partial differential equations with Riesz space fractional derivatives. In [7], new numerical methods was proposed. In [19,44], two-dimensional fractional model is discussed. These numerical methods tend to yield full coefficient matrices, which have computational cost of $O(K^3)$ and storage of $O(K^2)$ with K being the number of unknowns. To the best knowledge of the authors, there are few studies on the numerical treatment of RFADEs, efficient numerical methods, stability and convergence are still limited. This motivates us to investigate a more efficient numerical method for solving the RFADEs.

In this paper we present a unconditionally stable and fast solvable finite difference method to solve the RFADEs. We use an asymmetric technique introduced by Saluyev [31] to modify the shifted Grünwald's formula. This idea has been used in [28]. Formally the scheme is an implicit scheme, but by sequencing the nodal-points from one side to the other side and calculating the unknowns according to the sequences at the odd time-level and calculating the unknowns according to the opposite sequences at even time-level, the solution can be obtained explicitly, which is very convenient. We use the fast matrix-vector multiplication with the help of Toeplitz structure in our scheme, which has computational cost and storage of $O(K\log K)$. We prove that the error estimates in discrete l^2-norm between the numerical and analytical solutions is of order $O(\Delta t^2h^{-2(\alpha-1)}+\Delta t^2h^{-(\alpha+\epsilon-1)}+\Delta t+h)$, where h and Δt are the space and time meshsizes respectively, $\epsilon>0$ is any fixed constant. This means the error estimate is independent of the value of β. The result also shows that when $\alpha\in(1,1.5]$, the error estimate is optimal with $\Delta t=O(h)$, and when $\alpha\in[1.5,2)$, the error estimate is optimal with $\Delta t=O(h^{\alpha-0.5})$. From the result we can also conclude that the convergence rates in space are optimal. The result has the same order as the implicit shifted Grünwald finite difference scheme.

The asymmetric technique has been used to construct parallel algorithms by several researches, see, for example, [10,11,42,43]. But these papers just present the stability property, mentioning that the truncation error is of order $O(\Delta th^{-1}+\Delta t+h)$ for parabolic problems [10,11], or just present the asymmetric technique [42,43] in real calculations. [28] is the first paper to analyze the error for the asymmetric technique and get the result that the error between the discrete and the analytical solutions is of order $O\big(\Delta t^2h^{-2(\alpha-1)}\big)$ for fractional-order diffusion equations. This paper is a generalization of [28].

The rest of the paper is organized as follows. In Section 2, we present the scheme of the finite difference methods by using the asymmetric technique. In Section 3, we prove the solvability and unconditional stability of the scheme and give the computational cost for the scheme at each time step. The error estimates in discrete l^2-norm are presented in Section 4. In Section 5, some numerical examples are presented to verify the theoretical appearances.

2. The asymmetric finite difference scheme

In this section, we present the numerical scheme for the one-dimensional initial-boundary value problem of the RFADE

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=K_\alpha\dfrac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}+K_\beta\dfrac{\partial^\beta u(x)}{\partial|x|^\beta}+f(x,t), & L\le x\le R, 0\le t\le T,\\ u(x=L,t)=0, u(x=R,t)=0, & 0\le t\le T,\\ u(x,t=0)=u_0(x), & L\le x\le R,\end{cases}\tag{1}$$

where K_α and K_β represent the dispersion coefficient and the average fluid velocity , $u_0(x)$ is continuous on $[L, R]$, $f(x, t)$ is continuous on $[L, R]\times[0, T]$.

The Riesz space fractional derivatives $\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}$ and $\frac{\partial^\beta u(x)}{\partial|x|^\beta}$ of order $1<\alpha\le 2$ and $0<\beta\le 1$ are defined respectively, as[41]

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}=-c_\alpha({}_LD_x^\alpha+{}_xD_R^\alpha)u(x,t),\tag{2}$$

$$\frac{\partial^\beta u(x,t)}{\partial|x|^\beta}=-c_\beta({}_LD_x^\beta+{}_xD_R^\beta)u(x,t),\tag{3}$$

where $c_\alpha=\frac{1}{2\cos(\frac{\pi\alpha}{2})}<0$, $c_\beta=\frac{1}{2\cos(\frac{\pi\beta}{2})}>0$, $\beta\ne 1$. ${}_LD_x^\alpha$, ${}_xD_R^\alpha$, ${}_LD_x^\beta$ and ${}_xD_R^\beta$ are defined as the left- and right- Riemman–Liouville fractional derivatives [27].

In the case of $\beta=1$ and $\alpha=2$, the above equation reduces to the classical ADE. We assume that $K_\alpha>0$, $K_\beta\ge 0$, so that the flow is from left to right.

Let $h=(R-L)/K$ be the spatial meshsize and $\Delta t=T/2N$ be the time increment, where K and N are positive integers.

For $n=0,1,2,\dots,2N$ and $l=0,1,2,\dots,K$, we give some notations,

$$t^n=n\Delta t,\quad x_l=L+lh,\quad u_l^n=u(x_l,t^n),\quad f_l^n=f(x_l,t^n).$$