Applied Mathematics and Computation 332 (2018) 228–240

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Influence of complex coefficients on the stability of difference scheme for parabolic equations with non-local conditions

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ARTICLE INFO

Keywords:
Non-local boundary condition
Complex coefficients
Eigenvalues of finite difference operator
Parabolic equation
Kuramoto–Tsuzuki equation
Stability of finite difference scheme

ABSTRACT

The *stability* of a finite difference scheme for Schrödinger, Kuramoto–Tsuzuki and parabolic equations, subject to *non-local* conditions with *complex* coefficients, is dealt with. The stability conditions, which have to be met by complex coefficients in non-local conditions, have been determined. The main result of this study is that complex coefficients together with non-local conditions cause new effects on the stability of difference scheme. Numerical experiment has revealed additional regularities in the stability conditions.

1. Introduction

During several recent decades much attention in mathematical literature has been paid to numerical methods of various types of differential equations with *non-local* conditions and their applications (see [1,5,10,11,17–19,28,29] and references therein). On the other hand, there are many differential equations with the classical conditions where we can distinguish the problems with complex coefficients both in the theoretical and application sense (Schrödinger, Kuramoto–Tsuzuki and similar equations). Schrödinger equations with complex coefficients in boundary conditions have also been considered [2,30,31].

These facts gave us a motive to study the influence of *complex* coefficients in non-local conditions on numerical methods.

Other authors have already dealt with similar problems. As for instance, the structure of the spectrum of the second order differential operator, dependent on a complex parameter in non-local conditions, has been investigated. We are going to briefly review a few of such studies.

In [20], the eigenvalue problem with various non-local conditions has been investigated, as a separate case aiming at the eigenvalue problem

$$
\begin{aligned}
&-u'' = \lambda u, \qquad t \in (0,1),\\
&u(0) = 0,\\
&u(1) = \gamma \int_{\xi}^{1} u(t)\,dt, \qquad \text{or} \qquad u(1) = \gamma \int_{0}^{\xi} u(t)\,dt,
\end{aligned}
$$

where $\xi \in (0,1)$, γ is a real or complex number. The issues of complex eigenvalues existence with varying ξ and γ have been examined in detail in the discussed work. The cases where all eigenvalues are real have been explored. A similar problem has also been considered in [27].

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https://doi.org/10.1016/j.amc.2018.03.072
0096-3003/

Original research results on the spectrum of differential as well as difference operators (containing a complex parameter in non-local condition) have been obtained in [8,9] – in connection with investigation the stability of difference schemes for differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in (0,1),$$

subject to the conditions

$$u(x,0) = u_0(x), \qquad u(0,t) = 0, \qquad \gamma \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x},$$

where γ is a complex parameter. In this paper, the authors use the concept of neutral stability curve to investigate the stability of difference schemes. This curve separates the stability ant instability regions. At each point of the curve there exists one eigenvalue λ with the property $\mathrm{Re}\,\lambda = 0$. We shall also use this concept in our paper. The authors of [16] have investigated the stability of difference schemes for Kuramoto–Tsuzuki equation subject to non-local integral-type boundary conditions.

The main result proved by us in this article is that complex coefficients together with non-local conditions cause new effects – influencing the stability of difference schemes. We shall also present the results of a numerical experiment, revealing additional regularities in the stability conditions.

The rest of this paper is organized as follows. In Section 2, we formulate the differential and difference problems; the latter will be expressed in the standard form (as a two-layer difference scheme). In Section 3, we briefly discuss the stability of difference schemes in a specially selected energetic norm. The impact of a complex coefficient *in the integral-type boundary conditions* on the structure of the spectrum of the difference operator is dealt with in Section 4, and the stability conditions of the difference scheme are stated in Section 5. In Section 6, we provide short analysis of the influence of a complex coefficient *in the differential equation* on the spectrum of the difference operator, addressing the stability as well. The results of the numerical experiment and conclusions are presented in Section 7.

2. Statement of the problem

We consider the parabolic equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad x \in (0,1), \qquad t \in (0,T], \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x), \qquad x \in [0,1], \tag{2}$$

and non-local integral-type boundary conditions

$$u(0,t) = \gamma_1 \int_0^1 \alpha(x)\,u(x,t)\,dx + \mu_1(t), \qquad t > 0, \tag{3}$$

$$u(1,t) = \gamma_2 \int_0^1 \beta(x)\,u(x,t)\,dx + \mu_2(t), \qquad t > 0, \tag{4}$$

where f, φ, μ_1, μ_2, α, β are known functions and c, γ_1, γ_2 are given numbers. The specificity of problem (1)–(4) is conditioned by the fact that the parameters c, γ_1 and γ_2 can be *complex* numbers. We assume that $\alpha(x)$ and $\beta(x)$ are real-valued functions.

To solve the differential problem (1)–(4) numerically, one can apply an implicit difference scheme

$$\frac{u_k^{n+1} - u_k^n}{\tau} = c\,\frac{u_{k-1}^{n+1} - 2u_k^{n+1} + u_{k+1}^{n+1}}{h^2} + f_k^{n+1}, \qquad k = 1, 2, \dots, N-1, \tag{5}$$

$$u_k^0 = \varphi_k, \qquad k = 0, 1, \dots, N, \tag{6}$$

$$u_0^{n+1} = \gamma_1 h\left(\frac{\alpha_0 u_0^{n+1} + \alpha_N u_N^{n+1}}{2} + \sum_{k=1}^{N-1} \alpha_k u_k^{n+1}\right) + (\mu_1)^{n+1}, \tag{7}$$

$$u_N^{n+1} = \gamma_2 h\left(\frac{\beta_0 u_0^{n+1} + \beta_N u_N^{n+1}}{2} + \sum_{k=1}^{N-1} \beta_k u_k^{n+1}\right) + (\mu_2)^{n+1}, \tag{8}$$