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Fast algorithms for computing the characteristic polynomial of threshold and chain graphs

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ABSTRACT

The characteristic polynomial of a graph is the characteristic polynomial of its adjacency matrix. Finding efficient algorithms for computing characteristic polynomial of graphs is an active area of research and for some graph classes, like threshold graphs, there exist very fast algorithms which exploit combinatorial structure of the graphs. In this paper, we put forward some novel ideas based on divisor technique to obtain fast algorithms for computing the characteristic polynomial of threshold and chain graphs.

1. Introduction

The characteristic polynomial of a graph is the characteristic polynomial of its adjacency matrix. A lot of work is devoted in spectral graph theory for finding fast algorithms for computing the characteristic polynomial of graphs, especially those graphs with some distinguished structure. For example, threshold and chain graphs are relatively sparse and can have "rich" automorphism groups, and this makes it possible to speed up computations of their characteristic polynomials.

In this sense, recent papers of Jacobs et al. [17] and Fürer [14] exploit combinatorial structure of the adjacency matrix of a threshold graph to get fast algorithms computing its characteristic polynomial. The authors of [17] find first a diagonal matrix congruent to the matrix $A - xI$, where A is the adjacency matrix of a threshold graph of order n, and then compute the characteristic polynomial as a product of diagonal elements in $O(n^2 \log n)$ time. This running time is reduced to $O(n \log^2 n)$ in [14], where a recurrence relation for the characteristic polynomial of a threshold graph is iteratively computed in rounds of pairwise 2×2 matrix multiplications.

In this paper, we combine the divisor technique and well-known Schwenk-like formula (see, e.g. [11,12]) to devise fast algorithms for computing the characteristic polynomial of threshold and chain graphs. The worst-case running time of our algorithms is comparable to the asymptotic bound in [14], but our algorithms are simpler and easier to implement, and in some cases they run in linear time.

The rest of the paper is organized as follows. In Section 2 we briefly review relevant definitions and results which are subsequently needed in the paper. In Section 3 we consider threshold graphs and present a fast algorithm for computing

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their characteristic polynomial. Section 4 deals with chain graphs in an analogous way and so we therein omit a detailed discussion of obvious modifications. Finally, in Section 5 we add some concluding remarks regarding possible generalizations of our approach.

2. Preliminaries

We consider mainly simple graphs, i.e., finite undirected graphs without loops or multiple edges. However, in order to describe our algorithms, we also need digraphs and weighted (di)graphs, and occasionally allow loops.

As usual, in (di)graph $G = (V, E)$, V is its vertex set and E is its edge set; $n = |V|$ is the order of G and $m = |E|$ is the size of G. A weighted (di)graph $\widehat{G} = (G, w)$ consists of its underlying graph $G = (V, E)$ endowed by the weight function $w : E \to \mathbb{R}$. If $uv \notin E$ we assume that $w(uv) = 0$; unweighted (di)graphs can be regarded as weighted with w being the constant 1. The square matrix $A(\widehat{G}) = [a_{uv}]$ of order n with $a_{uv} = w(uv)$ is the weighted adjacency matrix of $\widehat{G}$. The adjacency matrix of an unweighted (di)graph is the usual (0, 1)-matrix.

The polynomial $\phi(x; \widehat{G}) = \det(xI - A(\widehat{G}))$ is the characteristic polynomial of $\widehat{G}$. Clearly, it is an invariant of $\widehat{G}$, that is, it does not depend of vertex labeling. We postulate that $\phi(x; \emptyset) = 1$, where $\emptyset$ stands for empty graph, i.e., vertex-free graph.

If $U \subset V$, then $G - U$ is the subgraph (or subdigraph) of the (di)graph G obtained by deleting all vertices from U together with (directed) edges incident to vertices from U. If $v \in V$, we also write $G - v$ for $G - \{v\}$. Similar notation is used for weighted (di)graphs, keeping in mind that the weight function then becomes the restriction from V to $V \setminus U$.

Given a simple graph $G = (V, E)$, a partition $\pi(G)$ of its vertex set V into mutually disjoint and non-empty cells $V_1, V_2, \ldots, V_k$ such that $V = V_1 \cup V_2 \cup \cdots \cup V_k$ is an *equitable partition* of G if for any ordered pair (i, j), $(1 \le i, j \le k)$, there is a number b_{ij} such that every vertex $v \in V_i$ has exactly b_{ij} neighbors in V_j. This gives rise to the $k \times k$ matrix $B = \left[b_{ij}\right]$, which can be considered as an adjacency matrix of a weighted digraph $\widehat{D} = (D, w)$, where

- The vertex set of D is $\{v_1, v_2, \ldots, v_k\}$, where each v_i corresponds (in one-to-one fashion) to the cell V_i, $(1 \le i \le k)$, of $\pi(G)$;
- The edge set of D contains directed edges v_iv_j if $i \neq j$, or loops v_iv_j if $i = j$;
- The weight function w is defined by $w(v_iv_j) = b_{ij}$, $(1 \le i, j \le k)$.

The weighted digraph $\widehat{D}$ is called the *divisor* of G with respect to $\pi(G)$. Note that a graph G can have more than one divisor. (A trivial one arises if each of its vertices constitute one cell.) The most important property of divisors is that the characteristic polynomial of any divisor divides the characteristic polynomial of the original graph, i.e., $\phi(x; \widehat{D})$ divides $\phi(x; G)$.

Because the most general Schwenk's formula (see, e.g. [2]) for computing the characteristic polynomial of weighted graphs is more than what we need here, we state only one of its variants which is sufficient for our purposes.

Theorem 2.1. *Let $A = \left[a_{ij}\right]$ be a symmetric matrix, and let $\widehat{G}$ be the corresponding weighted graph. If u is a pendant vertex (i.e., a vertex of degree one) of $\widehat{G}$ with the unique neighbor v, then*

$$\phi(x; \widehat{G}) = (x - a_{uu})\phi(x; \widehat{G} - u) - a_{uv}^2\phi(x; \widehat{G} - u - v), \tag{2.1}$$

where $\widehat{G} - u$ and $\widehat{G} - u - v$ are graphs obtained by deleting the vertex u or the vertices u and v, respectively, from $\widehat{G}$.

3. Threshold graphs

Threshold graphs were introduced by Chvatal and Hammer (see [9,10]), and later rediscovered by different authors in different contexts. Many real life phenomena in physics, biology, social sciences, etc. can be modeled by threshold graphs. In spectral graph theory, they are known as graphs which admit the "step-wise form of the adjacency matrix" or "nested split graphs" (see [6] and [12], respectively). In [19], it was observed that the graphs with maximal largest eigenvalue within (connected) graphs of fixed order and size are $\{2K_2, P_4, C_4\}$-free graphs,[1] and consequently threshold graphs. For more details about threshold graphs readers are referred to [7,18].

In this paper we use a very convenient bijection between binary sequences of length $n - 1$ and threshold graphs on n vertices, so that a threshold graph is identified by a binary sequence. This characterization follows from the following construction of threshold graphs: it starts with a single vertex, and proceeds by adding sequentially at each step either a dominating vertex or an isolated vertex. More precisely, for a given binary sequence $b = b_1b_2 \ldots b_n$, $(b_i \in \{0, 1\})$, the associated threshold graph $G(b)$ is constructed as follows:

(i) for $i = 1$, $G_1 = G(b_1) = K_1$, i.e., a single vertex;
(ii) for $i = 2, \ldots, n$, with $G_{i-1} = G(b_1 \ldots b_{i-1})$ already constructed, $G_i = G(b_1 \ldots b_{i-1}b_i)$ is formed by adding an isolated vertex to G_{i-1} if $b_i = 0$ (that is, a vertex non-adjacent to any vertex in G_{i-1}), or by adding a dominating vertex to G_{i-1} if $b_i = 1$ (that is, a vertex adjacent to all vertices in G_{i-1}).

[1] P_n, C_n and K_n stand for a path, cycle and complete graph of order n, respectively.