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Heat balance integral methods applied to the one-phase Stefan problem with a convective boundary condition at the fixed face

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ABSTRACT

In this paper we consider a one-dimensional one-phase Stefan problem corresponding to the solidification process of a semi-infinite material with a convective boundary condition at the fixed face. The exact solution of this problem, available recently in the literature, enable us to test the accuracy of the approximate solutions obtained by applying the classical technique of the heat balance integral method and the refined integral method, assuming a quadratic temperature profile in space. We develop variations of these methods which turn out to be optimal in some cases. Throughout this paper, a dimensionless analysis is carried out by using the parameters: Stefan number (Ste) and the generalized Biot number (Bi). In addition it is studied the case when Bi goes to infinity, recovering the approximate solutions when a Dirichlet condition is imposed at the fixed face. Some numerical simulations are provided in order to estimate the errors committed by each approach for the corresponding free boundary and temperature profiles.

1. Introduction

Stefan problems model heat transfer processes that involve a change of phase. They constitute a broad field of study since they arise in a great number of mathematical and industrial significance problems [1–4]. A large bibliography on the subject was given in [5] and a review on analytical solutions is given in [6].

Due to the non-linearity nature of this type of problems, exact solutions are limited to a few cases and it is necessary to solve them either numerically or approximately. The heat balance integral method introduced by Goodman in [7] is a well-known approximate mathematical technique for solving heat transfer problems and particularly the location of the free boundary in heat-conduction problems involving a phase of change. This method consists in transforming the heat equation into an ordinary differential equation over time by assuming a quadratic temperature profile in space. In [8–14] this method is applied using different accurate temperature profiles such as: exponential, potential, etc. Different alternative pathways to develop the heat balance integral method were established in [15].

In the last few years, a series of papers devoted to integral method applied to a variety of thermal and moving boundary problems have been published [16–19]. The recent principle problem emerging in application in heat balance integral method has spread over the area of the non-linear heat conduction: [20–22], and fractional diffusion: [23–25].

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In this paper, we obtain approximate solutions through integral heat balance methods and variants obtained thereof proposed in [15] for the solidification of a semi-infinite material $x > 0$ when a convective boundary condition is imposed at the fixed face $x = 0$.

The convective condition states that heat flux at the fixed face is proportional to the difference between the material temperature and the neighbourhood temperature, i.e.:

$$k\frac{\partial T}{\partial x}(0,t) = H(t)(T(0,t) + \Theta_\infty),$$

where T is the material temperature, k is the thermal conductivity, $H(t)$ characterizes the heat transfer at the fixed face and $-\Theta_\infty < 0$ represents the neighbourhood temperature at $x = 0$.

In this paper we will consider the solidification process of a semi-infinite material when a convective condition at the fixed face $x = 0$ of the form $H(t) = \frac{h}{\sqrt{t}}, h > 0$ is imposed. There are very few research studies that examine the heat balance integral method applied to Stefan problems with a convective boundary condition. Only in [7,26,27], a convective condition fixing $H(t) = h > 0$ is considered. Although approximate solutions are provided, their precision is verified by making comparisons with numerical methods since there is not exact solution for this choice of H.

The mathematical formulation of the problem under study with its corresponding exact solution given in [28] will be presented in Section 2. Section 3 introduces approximate solutions using the heat balance integral method, the refined heat balance integral method and two alternatives methods for them. In Section 4 we also study the limiting cases of the obtained approximate solutions when $h \to \infty$, recovering the approximate solutions when a temperature condition at the fixed face is imposed. Finally, in Section 5 we compare the approximate solutions with the exact one to the problem presented in Section 1, analysing the committed error in each case.

2. Mathematical formulation and exact solution

We consider a one-dimensional one-phase Stefan problem for the solidification of a semi-infinite material $x > 0$, where a convective condition at the fixed face $x = 0$ is imposed. This problem can be formulated mathematically in the following way:

Problem (P). Find the temperature $T = T(x, t)$ at the solid region $0 < x < s(t)$ and the location of the free boundary $x = s(t)$ such that:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0, \tag{1}$$

$$k\frac{\partial T}{\partial x}(0,t) = \frac{h}{\sqrt{t}}(T(0,t) + \Theta_\infty), t > 0, \tag{2}$$

$$T(s(t),t) = 0, t > 0, \tag{3}$$

$$k\frac{\partial T}{\partial x}(s(t),t) = \rho\lambda\dot{s}(t), t > 0, \tag{4}$$

$$s(0) = 0\ . \tag{5}$$

where the thermal conductivity k, the mass density ρ, the specific heat c and the latent heat per unit mass λ are given positive constants. The condition (2) represents the convective condition at the fixed face where $-\Theta_\infty < 0$ is the neighbourhood temperature at $x = 0$ and $h > 0$ is the coefficient that characterizes the heat transfer at the fixed face.

The analytical solution for the problem (P), using the similarity technique, was obtained in [28] and for the one-phase case is given by:

$$T(x,t) = -A\Theta_\infty + B\Theta_\infty \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right), \tag{6}$$

$$s(t) = 2\xi\sqrt{\alpha t}, \qquad \left(\alpha = \frac{k}{\rho c}: \text{ diffusion coefficient}\right) \tag{7}$$

where the constants A and B are defined by:

$$A = \frac{\operatorname{erf}(\xi)}{\frac{1}{\mathrm{Bi}\sqrt{\pi}} + \operatorname{erf}(\xi)}, \tag{8}$$