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Solving second-order linear differential equations with random analytic coefficients about ordinary points: A full probabilistic solution by the first probability density function

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ABSTRACT

This paper deals with the approximate computation of the first probability density function of the solution stochastic process to second-order linear differential equations with random analytic coefficients about ordinary points under very general hypotheses. Our approach is based on considering approximations of the solution stochastic process via truncated power series solution obtained from the random regular power series method together with the so-called Random Variable Transformation technique. The validity of the proposed method is shown through several illustrative examples.

1. Introduction and motivation

The aim of this paper is to provide a full probabilistic description, through the approximation of the first probability density function (1-p.d.f.), $f_1(x, t)$, of the solution stochastic process (s.p.), $X(t)$, to the second-order random linear differential equation

$$X''(t) + p(t; A)X'(t) + q(t; A)X(t) = 0, \quad t > t_0 \in \mathbb{R}, \tag{1}$$

with initial conditions

$$X(t_0) = Y_0, \quad X'(t_0) = Y_1. \tag{2}$$

In the initial value problem (IVP) (1) and (2), A, Y_0 and Y_1 are assumed to be absolutely continuous real random variables (r.v.'s) defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the sake of clarity, realizations for any r.v., say $Z : \Omega \longrightarrow \mathcal{D}_Z \subset \mathbb{R}$, will be denoted by $z(\omega) \in \mathcal{D}_Z$, $\omega \in \Omega$. As usual, from now on $\mathcal{D}_Z$ will be referred to as the range or codomain of Z. To provide more generality in our study, we will assume that these r.v.'s are statistically dependent, being $f_{A,Y_0,Y_1}(a, y_0, y_1)$ the joint probability density function (p.d.f.) of the random vector (A, Y_0, Y_1). The domain of this function will be denoted by $\mathcal{D}_{A,Y_0,Y_1} \subset \mathbb{R}^3$. For convenience, we also introduce the following notation, that will be used later, $\mathcal{D}_A \subset \mathbb{R}$, $\mathcal{D}_{Y_1} \subset \mathbb{R}$ and $\mathcal{D}_{A,Y_1} \subset \mathbb{R}^2$, that stand for the codomains of r.v.'s A and Y_1, and random vector (A, Y_1), respectively. Throughout

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this paper, we will assume that

$$\mathbf{H1}: \quad \begin{array}{l} f_{A,Y_0,Y_1}(a, y_0, y_1) \text{ is continuous in } y_0 \text{ and bounded, i.e.,} \\ \exists M_f > 0: \ |f_{A,Y_0,Y_1}(a, y_0, y_1)| < M_f, \ \forall (a_0, y_0, y_1) \in \mathcal{D}_{A,Y_0,Y_1}. \end{array} \tag{3}$$

It is remarkable to point out that a wide range of probabilistic distributions satisfy hypothesis H1, for example, gaussian, uniform, beta, gamma, log-normal, chi-square, t-student, etc. In particular, this condition is met for any multimodal distribution.

Henceforth, we will assume that coefficients $p(t;\ A)$ and $q(t;\ A)$ are s.p.'s, which depend on r.v. A, such that they are analytic about $(t_0;\ a_0(\omega))$ for every $a_0(\omega) \in \mathcal{D}_A,\ \omega \in \Omega$, i.e.,

$$\mathbf{H2}: \quad \begin{array}{l} \text{there exists a neighbourhood } \mathcal{N}_{p,q}(t_0; a_0(\omega)) \subset [t_0, +\infty[\times \mathcal{D}_A \text{ where} \\ \qquad p(t; A),\ q(t; A) \text{ are analytic } \forall a_0(\omega) \in \mathcal{D}_A,\ \omega \in \Omega. \end{array} \tag{4}$$

With the standard identification $p(t;\ A) \equiv p(t;\ a(\omega))$, $a(\omega) \in \mathcal{D}_A$, $\forall \omega \in \Omega$, we remember that the s.p. $p(t;\ A)$ is analytic about $(t_0,\ a_0(\omega))$, $a_0(\omega) \in \mathcal{D}_A$, $\forall \omega \in \Omega$, if the deterministic function $p(t;\ a(\omega))$ is analytic about $(t_0,\ a_0(\omega))$. The same can be said for the s.p. $q(t;\ A)$. To facilitate our subsequent analysis, hereinafter we will assume that $p(t;\ A)$ and $q(t;\ A)$ satisfy sufficient conditions in order to guarantee the IVP (1) and (2) has a unique solution.

Notice that in order to simplify notation, we have assumed that $p(t;\ A)$ and $q(t;\ A)$ are analytic in a common neighbourhood, $\mathcal{N}_{p,q}(t_0; a_0(\omega))$, which in practice will be defined by the intersection of the corresponding domains of analyticity of $p(t;\ A)$ and $q(t;\ A)$.

With the aim of motivating our study, below we point out two main facts. First, the interest of problem (1) and (2) that we are going to deal with and, secondly, the usefulness of computing the 1-p.d.f., $f_1(x,\ t)$. On the one hand, problems of the form (1) and (2) are important since many differential equations appearing in Mathematical Physics fall into this class. In this regard, it is worth pointing out that some recent contributions dealing with particular random differential equations of the form (1) and (2) can be found in [1–3], for example. In [1,2] authors give solutions of Hermite and Airy random differential equations, respectively, using the so-called mean square and mean fourth calculus [4,5]. In both contributions, approximations for the mean and the variance of the solution s.p. are given. In [3], these two statistical moments are computed for a wide range of second-order random linear differential equations taking advantage of homotopy analysis method. Recently, in [6] an analogous study has been conducted by applying the random differential transformation method developed in [7]. On the other hand, the computation of the 1-p.d.f. of the solution s.p. is advantageous since from it all one-dimensional statistical moments of the solution can be computed,

$$\mathbb{E}\big[(X(t))^k\big] = \int_{-\infty}^{+\infty} (x(t))^k f_1(x,t)\,\mathrm{d}x, \quad k = 1, 2, \ldots$$

Hence, the mean $\mu_X(t) = \mathbb{E}[X(t)]$ and the variance, $\sigma_X^2(t) = \mathbb{V}[X(t)] = \mathbb{E}\big[(X(t))^2\big] - (\mu_X(t))^2$, are easily obtained as particular cases. In addition, $f_1(x,\ t)$ allows us to compute the probability that the solution lies in specific sets of interest,

$$\mathbb{P}[\{\omega \in \Omega : a \le X(t)(\omega) \le b\}], \quad -\infty \le a < b \le +\infty.$$

In this paper, the Random Variable Transformation (RVT) technique will be used to compute approximations of $f_1(x,\ t)$. For the sake of clarity in the presentation and notation, below we state a multidimensional version of RVT method.

Theorem 1 (Multidimensional Random Variable Transformation method). *[5, p.25]. Let us consider $\mathbf{X} = (X_1, \ldots, X_n)^\top$ and $\mathbf{Z} = (Z_1, \ldots, Z_n)^\top$ two n-dimensional absolutely continuous random vectors defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Let $\mathbf{r} : \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one deterministic transformation of $\mathbf{X}$ into $\mathbf{Z}$, i.e., $\mathbf{Z} = \mathbf{r}(\mathbf{X})$. Assume that $\mathbf{r}$ is continuous in $\mathbf{X}$ and has continuous partial derivatives with respect to each X_i, $1 \le i \le n$. Then, if $f_{\mathbf{X}}(\mathbf{x})$ denotes the joint probability density function of random vector $\mathbf{X}$, and $\mathbf{s} = \mathbf{r}^{-1} = (s_1(z_1, \ldots, z_n), \ldots, s_n(z_1, \ldots, z_n))^\top$ represents the inverse mapping of $\mathbf{r} = (r_1(x_1, \ldots, x_n), \ldots, r_n(x_1, \ldots, x_n))^\top$, the joint probability density function of random vector $\mathbf{Z}$ is given by*

$$f_{\mathbf{Z}}(\mathbf{z}) = f_{\mathbf{X}}(\mathbf{s}(\mathbf{z}))|J|, \tag{5}$$

where $|J|$, which is assumed to be different from zero, is the absolute value of the Jacobian defined by the determinant

$$J = \det\left(\frac{\partial \mathbf{s}^\top}{\partial \mathbf{z}}\right) = \det\begin{pmatrix} \dfrac{\partial s_1(z_1,\ldots,z_n)}{\partial z_1} & \cdots & \dfrac{\partial s_n(z_1,\ldots,z_n)}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial s_1(z_1,\ldots,z_n)}{\partial z_n} & \cdots & \dfrac{\partial s_n(z_1,\ldots,z_n)}{\partial z_n} \end{pmatrix}. \tag{6}$$

In the context of solving random ordinary/partial differential and difference equations, RVT method has been successfully applied to compute, both analytically and numerically, the 1-p.d.f. associated to the solution (see for example, [8–20]). To the best of our knowledge, the application of RVT technique has not been explored yet in the context that approximations are obtained via power series. The aim of this paper is computing approximations of the 1-p.d.f., $f_1(x,\ t)$, of the solution s.p.