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Metric-locating-dominating sets of graphs for constructing related subsets of vertices

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ABSTRACT

A dominating set S of a graph is a metric-locating-dominating set if each vertex of the graph is uniquely distinguished by its distances from the elements of S, and the minimum cardinality of such a set is called the metric-location-domination number. In this paper, we undertake a study that, in general graphs and specific families, relates metric-locating-dominating sets to other special sets: resolving sets, dominating sets, locating-dominating sets and doubly resolving sets. We first characterize the extremal trees of the bounds that naturally involve metric-location-domination number, metric dimension and domination number. Then, we prove that there is no polynomial upper bound on the location-domination number in terms of the metric-location-domination number, thus extending a result of Henning and Oellermann. Finally, we show different methods to transform metric-locating-dominating sets into locating-dominating sets and doubly resolving sets. Our methods produce new bounds on the minimum cardinalities of all those sets, some of them concerning parameters that have not been related so far.

1. Introduction

Metric-locating-dominating sets were introduced in 2004 by Henning and Oellermann [24] combining the usefulness of resolving sets, that roughly speaking differentiate the vertices of a graph, and dominating sets, which cover the whole vertex set. Resolving sets were defined in the 1970s by Slater [43], and independently by Harary and Melter [21], whereas dominating sets were introduced in the 1960s by Ore [37]. Both types of sets have received much attention in the literature because of their many and varied applications in other areas. For example, resolving sets serve as a tool for combinatorial optimization [39], game theory [20], and pharmaceutical chemistry [9]; and dominating sets are helpful to analyze computer networks [38], design codes [12], and model biological networks [23]. Although metric-locating-dominating sets are hard to handle, for entailing the complexity of the other two concepts, they have been studied in several papers, for instance [5,26,27], and further generalized in other works such as [35,44].

Let $G=(V(G),E(G))$ be a finite, simple, undirected, and connected graph of order $n=|V(G)|\geq 2$; the *distance* $d(u,v)$ between two vertices $u,v\in V(G)$ is the length of a shortest $u-v$ path. We say that a subset $S\subseteq V(G)$ is a *resolving set* of G if for every $x, y\in V(G)$ there is a vertex $u\in S$ such that $d(u, x)\neq d(u, y)$ (it is said that S *resolves* $\{x, y\}$), and the minimum cardinality of such a set is called the *metric dimension* of G, written as $\dim(G)$. See [1] for a survey on this well studied

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graph invariant. When S is also a *dominating set* of G (i.e., every $x \in V(G)\setminus S$ has a neighbor in S), then S is called a *metric-locating-dominating set* (MLD-set for short). The *metric-location-domination number* (resp., *domination number*), written as $\gamma_M(G)$ (resp., $\gamma(G)$), is the minimum cardinality of an MLD-set (resp., dominating set) of G. Concerning specifically $\gamma(G)$, the survey [22] provides fundamental results and major research achievements in problems related to this parameter.

This paper first focuses on the intrinsic relations among MLD-sets, resolving sets and dominating sets. Indeed, the corresponding parameters for all those sets satisfy by definition

$$\max\{\dim(G), \gamma(G)\} \le \gamma_M(G) \le \dim(G) + \gamma(G). \tag{1}$$

We consider here this chain restricted to trees; specifically, we characterize the trees for which equality occurs in (1), thereby continuing the work of Henning and Oellermann [24] that characterized the trees T with $\gamma_M(T) = \gamma(T)$. Analogous characterizations of trees in terms of other related invariants can be found in [3,19].

We also compare MLD-sets with other subsets of vertices defined by Slater [41] that are directly connected to them: the *locating-dominating sets*. They are dominating sets that distinguish vertices by using neighborhoods instead of distances. Locating-dominating sets are of interest for its applications; for instance, the authors of [18] have used them to approach a problem proposed by Boutin [4] that involves the metric dimension. Furthermore, locating-dominating sets have applications outside graph theory; among them: location of intruders in facilities [42], and detection of inoperable components in multiprocessor networks [5]. More formally, a *locating-dominating set* (LD-set for brevity) of G is a dominating set $S \subseteq V(G)$ such that $N(x)\cap S \neq N(y)\cap S$ for every $x, y \in V(G)\setminus S$. The minimum cardinality of such a set, denoted by $\gamma_L(G)$, is the *location-domination number* of G. There is also an extensive literature on $\gamma_L(G)$ studying multiple aspects: complexity [13,17], specific families [14,16,25,29,31], bounds [2,15,22,40], and approximation algorithms [45]. Clearly, an LD-set is an MLD-set, and so it is also a resolving set; consequently,

$$\dim(G) \le \gamma_M(G) \le \gamma_L(G). \tag{2}$$

See [5,27] for more properties of chain (2) and bounds concerning its three parameters. Regarding the relation between $\gamma_M(G)$ and $\gamma_L(G)$, we propose a way to obtain LD-sets from MLD-sets which helps us to extend the following result due to Henning and Oellermann.

Theorem 1.1 [24]**.** *For any tree T, it holds that $\gamma_L(T) < 2\gamma_M(T)$. However, there is no constant c such that $\gamma_L(G) \le c\gamma_M(G)$ for all graphs G.*

We finally find relationships between MLD-sets and other subsets for which, so far as we are aware, no direct connection is known: the *doubly resolving sets*. Cáceres et al. [6] introduced doubly resolving sets as a tool for computing the metric dimension of cartesian products of graphs and, following the same spirit, Hertz [28] used them for computing the metric dimension of some hypercubes. Furthermore, different authors have provided other interesting applications of them; for instance, in source location. Indeed, Chen and Wang [11] utilized doubly resolving sets for modeling the problem of locating the source of a diffusion in a complex network, which is necessary for controlling and preventing epidemic risks. See [8,10] for a similar approach and [30] for more information on general source location. Doubly resolving sets, that somehow distinguish vertices in two ways by means of distances, are formally defined as follows. Two vertices $u, v \in V(G)$ *doubly resolve* a pair $\{x, y\} \subseteq V(G)$ if $d(u,x) - d(u,y) \neq d(v,x) - d(v,y)$. A set $S \subseteq V(G)$ is a *doubly resolving set* of G if every pair $\{x, y\} \subseteq V(G)$ is doubly resolved by two vertices of S (it is said that S *doubly resolves* $\{x, y\}$), and the minimum cardinality of such a set is denoted by $\psi(G)$. Thus, a doubly resolving set is also a resolving set, and so

$$\dim(G) \le \psi(G). \tag{3}$$

Although it is not straightforward to deduce any relation between $\psi(G)$ and $\gamma_M(G)$ from their definitions, we provide here bounds on $\psi(G)$ in terms of $\gamma_M(G)$ by generating doubly resolving sets from MLD-sets. We thus obtain, for specific classes and general graphs, similar chains to expression (2) that include $\psi(G)$. For more references on $\psi(G)$ containing algorithmic studies and relations with other graph parameters, we refer the reader to [7,33,34] for results on specific families of graphs and [10,32,36] for results on general graphs.

The paper is organized as follows. In Section 2, we characterize all trees achieving the extremal values in expression (1). We then show in Sections 3 and 4 how to construct LD-sets and doubly resolving sets from MLD-sets in arbitrary graphs and specific families, thus producing bounds on the corresponding parameters. Specifically, we prove in Section 3 that $\gamma_L(G) \le \gamma_M^2(G)$ whenever G has no cycles of length 4 or 6 but, for arbitrary graphs, any upper bound on $\gamma_L(G)$ in terms of $\gamma_M(G)$ has at least exponential growth; in Section 4, we provide the bounds $\psi(G) \le \gamma_M(G)$ for graphs G with girth at least 5, and $\psi(G) \le \gamma_M(G) + \gamma(G)$ for any graph G. We conclude the paper with some remarks and open problems in Section 5.

2. MLD-sets of trees

Henning and Oellermann [24] provided a formula for the metric-location-domination number of trees and characterized the trees T with $\gamma_M(T) = \gamma(T)$, giving both results in terms of support vertices (see Theorem 2.1 below). Recall that a vertex u of a tree T is a *support vertex* whenever it is adjacent to some *leaf* (i.e., a vertex of degree 1), and it is a *strong support vertex* if there are two or more leaves adjacent to u. We denote by $\mathcal{S}(T)$ (resp., $\mathcal{S}'(T)$) the set of support (resp., strong support) vertices of T; $\ell'(T)$ is the number of leaves adjacent to a strong support vertex.