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On the extremal eccentric connectivity index of graphs

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ABSTRACT

For a graph $G=(V,E)$, the eccentric connectivity index of G, denoted by $\xi^c(G)$, is defined as $\xi^c(G)=\sum_{v\in V}\varepsilon(v)d(v)$, where $\varepsilon(v)$ and $d(v)$ are the eccentricity and the degree of v in G, respectively. In this paper, we first establish the sharp lower bound for the eccentric connectivity index in terms of the order and the minimum degree of a connected G, and characterize some extremal graphs, which generalize some known results. Secondly, we characterize the extremal trees having the maximum or minimum eccentric connectivity index for trees of order n with given degree sequence. Finally, we give a sharp lower bound for the eccentric connectivity index in terms of the order and the radius of a unicyclic G, and characterize all extremal graphs.

1. Introduction

All graphs considered in this paper are simple and undirected. Let $G=(V(G),E(G))$ be a graph with $e(G)=|E(G)|$. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v in G and $N[v]=N(v)\cup\{v\}$. The *degree* of v is $d(v)=|N(v)|$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of G, respectively. If $X\subseteq V(G)$, then $G[X]$ denotes the subgraph induced by X and $G-X=G[V(G)-X]$. For $Y\subseteq E(G)$, then $G-Y$ denotes the graph obtained from G by deleting all the edges in Y. If $X=\{v\}$ or $Y=\{uv\}$, we write $G-v$ or $G-uv$ for $G-X$ or $G-Y$, respectively. If $uv\notin E(G)$, then $G+uv$ is a graph obtained from G by adding an edge uv. The *distance* of two distinct vertices u,v, denoted by $d(u,v)$, is the length of a shortest (u,v)-path in G. The *eccentricity* $\varepsilon(v)$ of a vertex v is the distance between v and a furthest vertex from v. An *eccentric vertex* of a vertex v is a vertex furthest away from v. The *diameter* and *radius* of G, denoted by $diam(G)$ and $rad(G)$, are defined as the maximum and the minimum of the eccentricities in G, respectively. A *center* of a graph is a vertex whose eccentricity is equal to $rad(G)$. Denote by S_n, P_n and K_n, a star, a path and a complete graph of order n, respectively. A *rooted* tree T_u is a tree with a specified vertex u, called the root of T. A *caterpillar* is a tree in which the vertices of degree at least two induce a path. The *join* of two graphs G and H, denoted by $G+H$, is the graph obtained from the disjoint union $G\cup H$ by adding the edges $\{xy\,|\,x\in V(G),\ y\in V(H)\}$.

Let T be a caterpillar and $P=v_1v_2\dots v_k$ the path induced by the vertices of degree at least 2 in T. If $\min\{d(v_i),d(v_{k-i+1})\}\geq\max\{d(v_{i+1}),d(v_{k-i})\}$ for $1\leq i\leq\lceil\frac{k}{2}\rceil-1$, then we call T a *greedy caterpillar*. Let T be a tree with root u. The *height* of a vertex $v\in V(T)$ is $h_T(v)=d(u,v)$ and $h(T)=max\{h_T(v)\mid v\in V(T)\}$. The *level-degree sequence* of T is the list of multisets L_i which contains the degrees of vertices of height i. We call T a *level-greedy tree* if the vertices of T can be numbered such that if $h_T(v_i)=h_T(v_j)$, then $d(v_i)\geq d(v_j)$ if and only if $i<j$, and if $h_T(v_i)=h_T(v_j)=\ell\geq 2$ and v_i,v_j have distinct neighbors of height $\ell-1$, say $v_iv_{t_i},v_jv_{t_j}\in E(T)$ with $t_i<t_j$, then $d(v_i)\geq d(v_j)$. Furthermore, if T is a level-greedy tree such that $d(v_i)\geq d(v_j)$ if $h_T(v_i)<h_T(v_j)$, then we call T a *greedy tree*.

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A *unicyclic graph* is the one obtained by adding an edge to a tree. Let U_i^n be the graph of order n obtained from a cycle C_i by attaching $n-i$ pendent vertices to some vertex of the cycle. Let $\mathscr{G}_{2r-1}^n$ be the set of such graphs of order $n \geq 3r-1$: choose a maximum independent set I of a cycle C_{2r-1} and one vertex w of the nontrivial component of $C_{2r-1}-I$, attach one pendent vertex to each vertex of $I \cup \{w\}$, and the other vertices as pendent vertices are attached to the vertices on the cycle arbitrarily.

A graph G is called *k-connected* if $|V(G)| > k$ and $G-X$ is connected for any $X \subseteq V$ with $|X| < k$. The *connectivity* $\kappa(G)$ is the maximum value of k for which G is k-connected. A graph G is called *k-edge connected* if $e(G) > k$ and $G-Y$ is connected for any $Y \subseteq E(G)$ with $|Y| < k$. The *edge connectivity* $\lambda(G)$ is the maximum value of k for which G is k-edge connected.

The *eccentric connectivity index* of G, denoted by $\xi^c(G)$, introduced by Sharma et al. in [11], is defined as

$$\xi^c(G) = \sum_{v \in V} \varepsilon(v) d(v) = \sum_{uv \in E} \omega(uv),$$

where $\omega(uv) = \varepsilon(u) + \varepsilon(v)$ is called the weight of the edge uv.

As is known that the eccentric connectivity index has been used extensively in physical and biological properties, see [3,10]. It is also being used both in QSAR studies and in the environmental of hazard assessment chemicals, see [6]. Recently, Qi and Du [9] calculated similar indices concerning eccentricity for trees.

The eccentric connectivity index has been evaluated for some classes of graphs. Zhou and Du [13] and Morgan et al. [7] provided independently the sharp lower bound for eccentric connectivity index of a tree with given order and diameter. Morgan et al. [8] established the sharp lower bound for eccentric connectivity index of a connected graph with given diameter. The lower bound for $\xi^c(G)$ of any connected graph G was first studied by Zhou and Du [13] who showed that

Theorem 1. *$\xi^c(G) \geq 3n-3$ for any connected graph G of order $n \geq 4$ with equality if and only if $G \cong S_n$.*

Observe the extremal graph in Theorem 1 is of minimum degree one, and $\xi^c(G)$ depends on the degree of each vertex of G, one may think that $\xi^c(G)$ should have some relations with the minimum degree of a graph G. In this paper, our first goal is to extend Theorem 1 in terms of the order and the minimum degree of a graph G.

Theorem 2. *Let G be a connected graph of order $n \geq 2\delta^2 - 2\delta + 4$ with $\delta(G) = \delta$. Then $\xi^c(G) \geq (2\delta+1)n - 2\delta - 1$ for δ is odd or δ is even and n is odd with equality if and only if $\pi(G) = (n-1, \delta, \delta, \ldots, \delta)$, and $\xi^c(G) \geq (2\delta+1)n - 2\delta + 1$ for δ is even and n is even with equality if and only if $\pi(G) = (n-1, \delta+1, \delta, \ldots, \delta)$.*

For any connected graph of order at least 4 is of minimum degree at least one, we can see that Theorem 1 is a special case of Theorem 2.

Remark 1. The lower bound in Theorem 2 is attained which can be seen as follows. If $\delta = 1$, then $G = S_n$ takes the lower bound. If $\delta = 2$, then the lower bound can be attained by $G = K_1 + \frac{n-1}{2}K_2$ if n is odd and by $G = K_1 + (\frac{n-4}{2}K_2 \cup P_3)$ if n is even. If $\delta \geq 3$, then $G = K_1 + H_{\delta-1,n-1}$ takes the lower bound, where $H_{\delta-1,n-1}$ is a graph of order $n-1$, constructed by Harary in [4], with the following properties: $\kappa(H_{\delta-1,n-1}) = \delta(H_{\delta-1,n-1}) = \delta - 1$, $\pi(H_{\delta-1,n-1}) = (\delta-1, \delta-1, \ldots, \delta-1)$ if one of $\delta-1$ and $n-1$ is not odd and $\pi(H_{\delta-1,n-1}) = (\delta, \delta-1, \ldots, \delta-1)$ if both $\delta-1$ and $n-1$ are odd. Thus, G is a graph of order n with $\delta(G) = \kappa(G) = \delta$, $\pi(G) = (n-1, \delta, \delta, \ldots, \delta)$ if one of δ and n is not even and $\pi(G) = (n-1, \delta+1, \delta, \ldots, \delta)$ if both δ and n are even.

Noting that $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for any graph G, by Theorem 2 and Remark 1, we have the following corollaries.

Corollary 1. *Let G be a connected graph of order $n \geq 2\lambda^2 - 2\lambda + 4$ with $\lambda(G) = \lambda$. Then $\xi^c(G) \geq (2\lambda+1)n - 2\lambda - 1$ for λ is odd or λ is even and n is odd with equality if and only if $\pi(G) = (n-1, \lambda, \lambda, \ldots, \lambda)$, and $\xi^c(G) \geq (2\lambda+1)n - 2\lambda + 1$ for λ is even and n is even with equality if and only if $\pi(G) = (n-1, \lambda+1, \lambda, \ldots, \lambda)$.*

Corollary 2. *Let G be a connected graph of order $n \geq 2\kappa^2 - 2\kappa + 4$ with $\kappa(G) = \kappa$.*

Then $\xi^c(G) \geq (2\kappa+1)n - 2\kappa - 1$ for κ is odd or $\kappa \geq 4$ is even and n is odd with equality if and only if $\pi(G) = (n-1, \kappa, \kappa, \ldots, \kappa)$, and $\xi^c(G) \geq (2\kappa+1)n - 2\kappa + 1$ for $\kappa \geq 4$ is even and n is even with equality if and only if $\pi(G) = (n-1, \kappa+1, \kappa, \ldots, \kappa)$.

By the graphs given in Remark 1, the lower bounds in Corollaries 1 and 2 can be attained except the one in Corollary 2 when $\kappa(G) = 2$. In such a case, the lower bound in Corollary 2 can be improved as below.

Theorem 3. *Let G be a connected graph of order $n \geq 5$ with $\kappa(G) = 2$. Then $\xi^c(G) \geq 6n - 10$ with equality if and only if $G \cong K_2 + (n-2)K_1$ for $n \geq 6$ and $G \in \{C_5, K_2 + 3K_1\}$ for $n = 5$.*

Remark 2. By the definition of $\xi^c(G)$, we can see that Theorem 2 does not hold if G is a δ-regular graph with $diam(G) \leq 2$. It is well known that such a graph does not exist if $n > \delta^2 + 1$, and there do exist some δ-regular graphs of order n such that $n = \delta^2 + 1$ and $diam(G) = 2$. For $\delta = 2, 3$, C_5 and the Peterson graph as illustrated in Fig. 1 are the graphs with diameter 2 and $\delta^2 + 1$ vertices, respectively. This is to say that at least $n > \delta^2 + 1$ is necessary in Theorem 2. Hoffman and Singleton [5] showed that a δ-regular graph of order $\delta^2 + 1$ with diameter 2 can exist only if $\delta = 2, 3, 7$ or 57, which means that perhaps $n > \delta^2 + 1$ is not tight in general. On the other hand, if $\delta - 1$ is a prime power, then based on a result given in [12], one can construct a δ-regular graph with connectivity δ and diameter 2 of order $\delta^2 - \delta + 2$ if $\delta - 1$ is even or order