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The k-path vertex cover in Cartesian product graphs and complete bipartite graphs

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ABSTRACT

For a graph G and a positive integer k, a subset S of vertices of G is called a k-path vertex cover if S intersects all paths of order k in G. The cardinality of a minimum k-path vertex cover is denoted by $\psi_k(G)$, and called the k-path vertex cover number of G. In this paper, we study some Cartesian product graphs and give several estimations and the exact values of $\psi_k(G)$.

1. Introduction and motivation

All graphs in this paper are finite, undirected, without loops and multiple edges. We use $V(G)$, $E(G)$ to denote the vertex set and the edge set of graph G, respectively. For any integers $a<b$, let $[a, b]$ denote the set of integers $\{a, a+1, \dots, b\}$ for simplicity. For a real number x, let $\lfloor x \rfloor$, $\lceil x \rceil$ denote the maximum integer no more than x, and the minimum integer no less than x, respectively. For a positive integer k, a set $S \subseteq V(G)$ is called a k-path vertex cover (k-PVC) if every path on k vertices contains a vertex from S. The cardinality of a minimum k-path vertex cover is denoted by $\psi_k(G)$, and called the k-path vertex cover number of a graph G. We can get an upper bound of the value of $\psi_k(G)$ by constructing the minimal k-path vertex cover set S, where S is minimal if and only if for any $v \in S$, $F-v$ is not a k-PVC. In particular, we use the order of a path P to indicate the number of vertices on P while the number of edges of P is abbreviated as the length of a path.

The concept of k-path vertex cover was motivated by the problem related to the security protocols in wireless sensor networks in [4,6,7,12,13,23]. The essential features of all mentioned practical problems can be expressed in the language of graph theory. We use vertices to represent sensor devices, similar as the description, and use edges to represent communication channels between pairs of sensor devices. In [7], it is guaranteed that the data integrity on k-generalized Canvas Scheme can be considered as k-path vertex cover problem in real-life which guarantees at least one vertex is not captured by assailant on each path of length $k-1$. Therefore, we should protect at least one vertex on each k-path. Recently, motivated by [20], it is incorporated weight into consideration thought the cost of installing protected vertices are different. In some applications, the devices which are in one group of k-PVC can share the information each other. The problem also comes from the real world with the increasing traffic accidents which is caused by the increasing private cars. Therefore, we need to install cameras at pivotal crossings. Thus, we take the cost of cameras and installing fees into consideration to guarantee the lowest cost. Then it can be turned into the k-PVC problem [18]. Sometimes we require the subgraph induced by the set of k-PVC to be connected (one can see [20]).

We do not care $\psi_1(G)$ because $\psi_1(G)=|V(G)|$, so we always suppose that $2 \le k \le |V(G)|$ for $\psi_k(G)$ in our paper. It is shown that determining $\psi_k(G)$ is NP-hard for a general graph G and each $k \ge 2$, but polynomial for any tree [1]. It is obvious

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that the minimum vertex cover $\psi_2(G) = |V(G)| - \alpha(G)$, where $\alpha(G)$ is the independence number of graph G. The independence number of graph has been studied in [5,8,15,16]. In bipartite graphs, the perfect matching is introduced helping us to determine the $\psi_k(G)$ such that we can find a maximum matching in a bipartite graph by slightly modifying the procedure in [18]. An ordinary algorithm that determines such a matching in any graph has been given by Edmonds in [11].

Clearly, the value of $\psi_3(G)$ corresponds to the dissociation number, and the relation between them is $\psi_3(G) = |V(G)| - diss(G)$. A subset of vertices in a graph G is called a dissociation set if it induces a subgraph with maximum degree 1. We define the dissociation number of G that the number of vertices in a maximum cardinality dissociation set in G, which is abbreviated by $diss(G)$. The dissociation number was studied in [9,10,14], which was shown that determining the dissociation number of G is NP-hard in the class of bipartite graphs. It is clear that we can get the dissociation number of bipartite graphs through $diss(G) = |V(G)| - \psi_3(G)$ since $\psi_3(G)$ is not difficult to be calculated for them. Approximation algorithm for vertex cover P_3 problem, see [17–19,21,22], where Tu et al. introduced algorithms evaluating $diss(G)$ and $\psi_3(G)$ in them. In [23], the k-path vertex cover numbers of some Cartesian product graphs are obtained. Given a weight function w on the vertex set V, the minimum weight k-path vertex cover problem is to find a k-PVC of G with the minimum weight, which is called the minimum weight k-path vertex cover ($MWVCP_k$).

A wheel is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of the cycle. We denote the wheel with n vertices by W_n. Define the new vertex as v_0, and $\{v_0, v_1, \ldots, v_n\}$ is the vertex set of W_{n+1}. As usual, $K_{m,n}$, K_n, P_n and C_n denote the complete bipartite graph, complete graph, path and cycle of order n, respectively.

In this paper, we study the minimum cardinality of a k-PVC in some classes of graph products. The Cartesian product $G\Box H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1 = u_2$ and $v_1v_2 \in E(G)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$. The lexicographic product $G \circ H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The strong product $G \boxtimes H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set $V(G) \times V(H)$, and vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

It is obvious that the following results hold.

Lemma 1.1.

(1) *Let G be a simple and undirected graph. Then, for $2 \le k \le |V(G)|$, we have $\psi_k(G) \le \psi_{k-1}(G)$, and $\psi_s(G) \ge \psi_t(G)$ for any $t > s$ with $t, s \in [2, |V(G)|]$.*

(2) *If G' is a subgraph of G and $k \ge 2$, then $\psi_k(G) \ge \psi_k(G')$.*

Lemma 1.2. [3] *Let $k \ge 2$ and $n \ge k$ be positive integers. Then*

$$\psi_k(P_n) = \left\lfloor \frac{n}{k} \right\rfloor,$$

$$\psi_k(C_n) = \left\lceil \frac{n}{k} \right\rceil,$$

$$\psi_k(K_n) = n - k + 1.$$

Lemma 1.3. [2] *For each $k \ge 4$, $\psi_k(P_{2\lceil\sqrt{k}\rceil} \Box P_{3\lceil\sqrt{k}\rceil}) \ge \lceil\sqrt{k}\rceil$.*

Lemma 1.4. [1] *For every graph G without isolated vertices, we have*

$$\psi_k(G) \le |V(G)| - \frac{k-1}{k} \sum_{u \in V(G)} \frac{2}{1 + d(u)}.$$

In particular, $\psi_2(G) \le |V(G)| - \sum_{u \in V(G)} \frac{1}{1+d(u)}$.

Lemma 1.5. [2] *Let $k \ge 2$ and $d \ge k - 1$ be positive integers. Then, for any d-regular graph G, we have*

$$\psi_k(G) \ge \frac{d - k + 2}{2d - k + 2} |V(G)|.$$

2. Main results

The H-layer ${}^uH$ is the subgraph of $G\Box H$ induced by the vertices $\{(u, v), v \in V(H)\}$ for any fixed vertex $u \in V(G)$; analogously, for a fixed vertex $v \in V(H)$, the G-layer G^v is the subgraph of $G\Box H$ induced by the set of vertices $\{(u, v), u \in V(G)\}$. Clearly, any H-layer is isomorphic to H which is an induced subgraph of $G\Box H$. Moreover, any k-path vertex cover sets S of $G\Box H$ must contain at least $\psi_k(H)$ vertices in every H-layer.

Theorem 2.1. *Let $k \ge 4$ and $n \ge 3$ be positive integers, then*

$$\left\lceil \frac{4n}{k+1} \right\rceil \ge \psi_k(P_2 \Box C_n) \ge \begin{cases} 2\left\lfloor \frac{2n}{k+2} \right\rfloor, & \text{if } k \text{ is even}, \\ 2\left\lfloor \frac{2n}{k+1} \right\rfloor, & \text{if } k \text{ is odd}. \end{cases}$$