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Tetravalent half-arc-transitive graphs of order p^5

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ABSTRACT

A graph is *half-arc-transitive* if its full automorphism group acts transitively on vertices and edges, but not on arcs. Let p be a prime. It is known that there exists no tetravalent half-arc-transitive graph of order p or p^2. All the tetravalent half-arc-transitive graphs of order p^3 or p^4 have been classified in two previous papers [9,23]. As a continuation, in this paper, a classification is given of all tetravalent half-arc-transitive graphs of order p^5.

1. Introduction

Throughout this paper all graphs are finite, simple, connected and undirected unless specified otherwise. A graph Γ is said to be *vertex-transitive, edge-transitive,* or *arc-transitive* if its full automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on its vertex set $V(\Gamma)$, edge set $E(\Gamma)$ or arc-set $A(\Gamma)$, respectively. A graph is said to be *half-arc-transitive* provided that it is vertex-transitive, edge-transitive, but not arc-transitive.

Interest in half-arc-transitive graphs goes back to Tutte's work [20], 7.35, p.59 showing that the valency of a half-arc-transitive graph is necessarily even. In 1970, Bouwer [2] proved the existence of half-arc-transitive graphs of any given even valency at least 4. Since then, half-arc-transitive graphs have received a great deal of attention, and constructing and characterizing half-arc-transitive graphs is also currently an active topic in algebraic graph theory—see [1,7,8,10,15–18,24] for example.

Cayley graphs play an important role in the study of half-arc-transitive graphs. It is well-known that a graph Γ is a *Cayley graph* if $\mathrm{Aut}(\Gamma)$ has a group of automorphisms acting regularly on vertices (see [3], Lemma 16.3). Let G be a permutation group on a set Ω and $\alpha\in\Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha=1$ for every $\alpha\in\Omega$ and *regular* if G is transitive and semiregular.

Given a finite group G and an inverse closed subset $S\subseteq G\setminus\{1\}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on G with respect to S is a graph with vertex set G and edge set $\{\{g,sg\}|g\in G, s\in S\}$. A Cayley graph $\mathrm{Cay}(G,S)$ is connected if and only if S generates G. For $g\in G$, let $R(g)$ be the permutation of G defined by $R(g)$: $x\mapsto xg$ for $x\in G$. Set $R(G):=\{R(g)\,|\,g\in G\}$. It is well-known that $R(G)\le \mathrm{Aut}(\mathrm{Cay}(G,S))$, and so a Cayley graph is vertex-transitive. Let $\mathrm{Aut}(G,S)=\{\alpha\in \mathrm{Aut}(G)\mid S^\alpha=S\}$, where $\mathrm{Aut}(G)$ denotes the full automorphism group of G. In 1981, Godsil [12] proved that the normalizer of $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G,S))$ is $R(G)\rtimes \mathrm{Aut}(G,S)$. A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$ [21]. For convenience of the statement, in the remainder of this paper, we shall identify G with $R(G)$.

Let p be a prime. In 1993, Xu [22] gave a complete classification of tetravalent half-arc-transitive graphs of order p^3, and as a result, it was shown that the Holt's graph, the smallest half-arc-transitive graph constructed by Holt [11], is the unique

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tetravalent half-arc-transitive graph of order 27 up to graph isomorphism. Actually, Xu [22] obtained the classification by proving that every tetravalent half-arc-transitive graph of order p^3 is a normal Cayley graph on a metacyclic group. Inspired by this, much work has been done on construction or classification of half-arc-transitive Cayley graphs on metacyclic p-groups. For example, as a generalization of Xu's work [22], a classification is given of half-arc-transitive Cayley graphs on metacyclic p-groups of valency less than $2p$ in [14], where p is an odd prime, and actually, all such graphs are also normal Cayley graphs. In 2008, Feng et al. [9] classified the tetravalent half-arc-transitive graphs of order p^4, and the result also indicates that each of such graphs is a normal Cayley graph on a metacyclic group of order p^4.

Motivated by the work listed above, in this paper we attempt to give a classification of tetravalent half-arc-transitive graphs of order p^5. In [9], the classification of tetravalent half-arc-transitive graphs of order p^4 depends on the classification of groups of order p^4. However, this approach seems to be no longer valid when considering the tetravalent half-arc-transitive graphs of order p^5 since there are more than 60 groups of order p^5 (see, for example, [23]). Note that one key step in classifying such kinds of graphs is to prove that they are normal Cayley graphs in almost all cases. Actually, in this paper, we finish this key step by using the characterization of tetravalent edge-transitive Cayley graphs of order odd prime power given in [24], and then by using the normality of Cayley graphs, together with some group theoretical methods, we obtain the classification of tetravalent half-arc-transitive graphs of order p^5. The main result will be stated and proved in Section 3. As a byproduct, several infinite families of tetravalent half-arc-transitive Cayley graphs on non-metacyclic groups are constructed. This is quite different from the previous work in [9,22] because all tetravalent half-arc-transitive graphs of order p^3 or p^4 are Cayley graphs on metacyclic groups.

2. Preliminaries

In this paper, we shall give some preliminary results. We first fix some notations. For a positive integer n, the expression $\mathbb{Z}_n$ denotes the cyclic group of order n, $\mathbb{Z}_n^*$ denotes the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. For two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. Let G be a finite group. The full automorphism group and the center of G will be denoted by $\mathrm{Aut}(G)$ and $Z(G)$, respectively. For $x \in G$, denote by $o(x)$ the order of x.

For a group G and $a_1, a_2, \ldots, a_n \in G$, $n \geq 2$, the commutator $[a_1, a_2, \ldots, a_n]$ of $a_1, a_2, \ldots, a_n$ was recursively defined as follow: if $n = 2$, then $[a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2$; and if $n > 2$, then $[a_1, a_2, \ldots, a_n] = [[a_1, a_2, \ldots, a_{n-1}], a_n]$. Let

$$G_n = \langle [a_1, a_2, \ldots, a_n] \mid a_1, a_2, \ldots, a_n \in G \rangle.$$

Then $G_{n+1} \leq G_n$. It is not difficult to show that if $G = \langle M \rangle$ for some subset M of G, then

$$G_n = \langle [x_1, x_2, \ldots, x_n]^g \mid x_1, x_2, \ldots, x_n \in M, g \in G \rangle.$$

In particular, $G_2 = G'$ is just the derived subgroup of G.

The following proposition is a basic property of commutators and it is straightforward (see, for example, [19], 5.1.5).

Proposition 2.1. *Let G be a group. Then, for any $x, y, z \in G$, $[xy, z] = [x, z]^y[y, z]$ and $[x, yz] = [x, z][x, y]^z$.*

The *Frattini subgroup*, denoted by $\Phi(G)$, of a finite group G is defined to be the intersection of all maximal subgroups of G. Let p be a prime, let G be a finite p-group, and set $G^p = \langle g^p \mid g \in G \rangle$. The following theorem is called the *Burnside Basis Theorem*.

Proposition 2.2 *[19, Theorem 5.2.12]. Let G be a finite p-group with p a prime. Then, $\Phi(G) = G'G^p$. If $|G : \Phi(G)| = p^r$ then every generating set of G has a subset of r elements which also generates G. In particular, $G/\Phi(G) \cong \mathbb{Z}_p^r$.*

Let p be a prime. By Proposition 2.2, all minimal generating sets of a finite p-group G have the same cardinality, called the *rank* of G and denoted by $d(G)$.

A finite p-group G is called *p-abelian*, if for any $x, y \in G$, then $(xy)^p = x^py^p$.

Proposition 2.3 *[5, Theorem 3.1]. Let G be a finite p-group which is generated by two elements and whose derived subgroup G' is abelian. Then G is p-abelian if and only if the exponent of G' is no more than p and the nilpotent class $c(G) < p$.*

The following result is due to Burnside.

Proposition 2.4 [*6, p.241*]. *Let G be a finite p-group. If $Z(G')$ is cyclic, then G' is cyclic.*

For a graph Γ, we denote by $V(\Gamma)$ the set of all vertices of Γ, by $E(\Gamma)$ the set of all edges of Γ, by $A(\Gamma)$ the set of all arcs (ordered paries of adjacent vertices) of Γ, and by $\mathrm{Aut}(\Gamma)$ the full automorphism group of Γ. For $u, v \in V(\Gamma)$, denote by $\{u, v\}$ the edge incident to u and v in Γ.

Let Γ be a graph, and let X be a subgroup of $\mathrm{Aut}(\Gamma)$. If X is transitive on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$, then Γ is said to be *X-vertex-transitive*, *X-edge-transitive* or *X-arc-transitive*, respectively, and if X is transitive on $V(\Gamma)$ and $E(\Gamma)$ but intransitive on $A(\Gamma)$, then Γ is called *X-half-arc-transitive*. A graph Γ is called *half-arc-transitive* (resp. *vertex-transitive, edge-transitive, or arc-transitive*) if it is $\mathrm{Aut}(\Gamma)$-half-arc-transitive (resp. $\mathrm{Aut}(\Gamma)$-vertex-transitive, $\mathrm{Aut}(\Gamma)$-edge-transitive, or $\mathrm{Aut}(\Gamma)$-arc-transitive).