Applied Mathematics and Computation 331 (2018) 80–95

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A non-polynomial numerical scheme for fourth-order fractional diffusion-wave model

Xuhao Li, Patricia J.Y. Wong*

School of Electrical and Electronic Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore

ARTICLE INFO

Keywords:
Parametric spline
Quintic spline
Diffusion-wave equation
Fractional differential equation
Numerical solution

ABSTRACT

In this paper, we tackle the numerical treatment of a fourth-order fractional diffusion-wave problem. By using parametric quintic spline in the spatial dimension and an approximation of Caputo derivatives at half-points, we propose a numerical scheme and rigorously prove its solvability, convergence and stability in maximum norm. It is shown that the theoretical convergence order *improves* those of earlier work. To confirm, simulation is carried out to demonstrate the numerical efficiency of the proposed scheme as well as the better performance over other methods.

1. Introduction

Fractional derivatives and integrals have been widely used to model memory processes in mechanics, biochemistry, medicine, electrical engineering and others [2,3,25]. For example, fractional differential equations are used to model diffusion processes, see [22–24,27]. By replacing the first-order or second-order time derivative in the classical diffusion or wave equation by a fractional derivative of order $\gamma > 0$, a fractional diffusion equation describes, as γ increases from 0 to 2, processes of slow diffusion $(0 < \gamma < 1)$, classical diffusion $(\gamma = 1)$, diffusion-wave $(1 < \gamma < 2)$ and classical wave $(\gamma = 2)$. The classical diffusion or wave equation involves second-order space derivative, but as pointed out in [2,3,10,11,25,28,29], a *fourth-order* space derivative may be necessary in some models such as transverse vibration of beams and groove formation on the surface of metal by a grain boundary.

There has been some pioneer work on the analytical and numerical solutions of fourth-order fractional diffusion-wave models. Under appropriate assumptions, Agrawal [2,3] presented general solutions of fourth-order fractional diffusion-wave equations with zero nonlinearities in terms of Mittag-Leffler functions. There, the equation is transformed to frequency domain by taking finite sine transform with respect to the spatial variable and then Laplace transform in the temporal dimension. The analytical solution is obtained by first applying inverse Laplace transform and then inverse finite sine transform. In most cases, it is difficult if not impossible to find an explicit expression of the exact solution. Therefore, numerical methods with high accuracy are very useful and important in applications. Indeed, various numerical techniques [10,11,13–15,17,28,31] have been used successfully for fourth-order fractional diffusion equations. Specifically, Adomian decomposition method [17] and homotopy perturbation technique [10,11] have been proposed but without rigorous analysis. Both of these methods give a recursive formula under the assumption that the exact solution can be decomposed into a sum of infinite series. Recently, Siddiqi and Arshed [28] developed a quintic B-spline method for fourth-order partial differential equations with three types of boundary conditions – clamped, simply supported and transversely supported. However, only

* Corresponding author.
E-mail addresses: lixu0015@e.ntu.edu.sg (X. Li), ejywong@ntu.edu.sg (P.J.Y. Wong).

https://doi.org/10.1016/j.amc.2018.02.044
0096-3003/

error analysis for the semi-discrete scheme is given and no explicit spatial convergence order is established. For fourth-order diffusion-wave problems with Dirichlet boundary conditions, Hu and Zhang [13–15] have applied finite difference and compact finite difference methods to obtain spatial convergence order of $O(h^2)$ and $O(h^4)$, respectively, where h is the spatial grid size. They have proved that the numerical schemes are uniquely solvable, robust with respect to initial data and non-homogeneous data, and unconditionally convergent. Moreover, for fourth-order fractional sub-diffusion equations with Neumann boundary conditions, a higher order compact method ($O(h^4)$) has been discussed rigorously in [31]. Note that $O(h^4)$ is the *best spatial convergence* achieved to date in the literature of fractional diffusion problems.

Another key ingredient in the numerical treatment of fractional differential equations lies in the approximation of fractional derivatives. For fractional sub-diffusion equations (fractional order $0<\gamma<1$), Grünwald–Letnikov (GL) method [7], weighted shifted GL method [32], $L1$ discretization [9,14,20,31] and $L2-1_\sigma$ method [4] are mostly used to approximate fractional derivatives. For fractional diffusion-wave equations (fractional order $1<\gamma<2$), a widely used approximate scheme at half-points is developed in [30] via $L1$ discretization and Crank-Nicolson technique. Other than this formula, by using weighted shifted GL technique for fractional integral with fractional order $0<\gamma<1$, a more stable second order accuracy scheme can be obtained as in [16,32].

In the current work, we shall discuss the numerical treatment of a fourth-order fractional diffusion-wave problem which has also been considered in [13–15]. Noting that $O(h^4)$ is the *best spatial convergence* achieved in the literature, our aims are to improve the spatial convergence order to *above* 4, and to provide rigorous proof of solvability, convergence and stability of the proposed method. To achieve our objectives, we shall introduce a parametric quintic spline method in the spatial dimension, coupled with approximation of the fractional derivatives at half-points. It is noted that parametric quintic spline method has been effective in solving boundary value problems of *integer-order* differential equations [18,19], and subsequently parametric *cubic* spline has been applied to fractional differential equations [8,12], while the first application of parametric *quintic* spline to a *second-order* fractional sub-diffusion problem appears in [20]. We shall consider the following fourth-order fractional diffusion-wave problem [13–15]

$$ {}_0^C D_t^\gamma u(x,t) + b^2 \frac{\partial^4 u}{\partial x^4} = f(x,t), \qquad x\in(0,L), \quad t\in[0,T] \tag{1.1} $$

with initial conditions

$$ u(x,0)=\phi(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad x\in[0,L] \tag{1.2} $$

and boundary conditions

$$ u(0,t)=g_1(t), \quad u(L,t)=g_2(t), \quad \frac{\partial^2 u(0,t)}{\partial x^2} = g_3(t), \quad \frac{\partial^2 u(L,t)}{\partial x^2} = g_4(t), \quad t\in[0,T]. \tag{1.3} $$

Here, $1<\gamma<2$, L, T and b are positive constants, $f(x, t)$, $\phi(x)$, $\psi(x)$, $g_1(t)$, $g_2(t)$, $g_3(t)$ and $g_4(t)$ are known continuous functions with $\phi(0)=g_1(0)$ and $\phi(L)=g_2(0)$, and the Caputo fractional derivative ${}_0^C D_t^\gamma u(x,t)$ is defined by

$$ {}_0^C D_t^\gamma u(x,t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} (t-s)^{1-\gamma} ds. $$

Finally, we note that there are some works on the numerical treatment of other fractional problems (different from (1.1)–(1.3)) using polynomial methods in both temporal and spatial dimensions [5,6,33]. In [5], to solve a multi-term time-space fractional problem, the authors have proposed a space-time spectral algorithm based on shifted Jacobi tau technique combined with the associated operational matrices of various fractional derivatives and integrals. Based on a similar idea, in [6] new variable-order fractional (V-OF) operational matrices for the shifted Jacobi polynomials have been constructed, then a V-OF Schrödinger equation is solved using Jacobi collocation approach, with both temporal and spatial discretizations handled spectrally. Further, a class of distributed-order fractional optimal control problem can also be solved effectively using polynomial based numerical method, refer to [33] for details. As stated in [5,6,33], the usage of polynomials in both temporal and spatial dimensions can greatly reduce the storage, achieve excellent numerical accuracy and provide smooth approximate solutions. Indeed, all these works [5,6,33] will result in systems of linear or nonlinear algebraic equations that are solvable. It is noted that much computations are required to derive the algebraic equations, and the coefficient matrices are normally dense which are not easy to handle when they become large. Moreover, it is very challenging to give theoretical proof of the stability and convergence of the numerical schemes. On the other hand, in the present work we shall apply typical discretization in the temporal dimension and employ parametric quintic spline in the spatial dimension. This leads to a system of linear equations with a very sparse pentadiagonal coefficient matrix, which can be solved efficiently. Our proposed scheme obtains the numerical solutions by time levels and so may require larger storage compared to [5,6,33], but the scheme is rather simple and fast to implement in each iteration. Moreover, we are able to prove *theoretically* in subsequent sections that the proposed scheme is stable and achieves better accuracy than some previous work.

The paper is organized as follows. In Section 2, we derive the numerical scheme using parametric quintic spline operator and a well known approximation of Caputo derivatives at half-points. Then, in Sections 3 and 4, the solvability, stability and convergence of the proposed scheme are established rigorously by energy method. To confirm the theoretical results and to demonstrate the efficiency of the proposed scheme, numerical simulations are presented in Section 5. Finally, we end with a brief conclusion in Section 6.